The Juice Seller's Problem

Hello and welcome, I'm Ghada Suleiman Abdullah Marmash, a teacher in the schools of King Abdullah II, His Excellence from Jordan. I hope that you can help me solve a problem; this problem will develop your skills in mathematics and it requires some space and some knowledge of volumes. The juice seller faces this problem every day, since he is always trying to pour liquid from two containers into a third one, and he needs to know this without wasting time or effort. Let's try to help him.

Look .. . These are the containers holding the juice, and he wants to pour them into a third container. Unfortunately, he has no knowledge of the mathematics involved and tries to pour them into the third container . .. let's see. He could not pour the containers into the third one as he is trying for a third time, let's see the third container . . . Look at the problem! He could not fill this container. He will try a third time. Look! He finally succeeded, but it took a while .

This is a problem! He needs to know the solution before pouring the containers into the third one, into the required container. How can we help him? How can we help him save time, without knowledge of the mathematics involved? He does not know anything about mathematics. Let's think about this problem, the problem of finding the appropriate volume for pouring two containers into a third one without effort and without calculations . . . Thank you! Think about it a bit, I will see you shortly.

Hello again! Thank you for your cooperation in trying to solve the problem using the Pythagorean Theory, which we have mentioned before. It states that "the area of the square drawn on the hypotenuse of the right triangle equals the sum of the areas of the two squares drawn on the sides of the right angle". Let's play a game together to verify this theory. Note, this is the square drawn on one side of the right angle, we will try to dump the squares into the larger one, look . . .

This proves the Pythagorean Theory, where we were able to take the squares established on the sides of the right angle and empty them out into the larger square. Specifically, the area of the two squares drawn is equal to the area of the square drawn on the hypotenuse of the right angle.

Now after we have played this game, the question is: does the Pythagorean Theory remain true if we replace the squares with triangles, rectangles, hexagons or pentagons? Discuss that together and I will see you shortly.

Hello! You must have noticed that the theory remains valid if the shapes drawn on the sides of the right triangle are regular, i. e. they have the same lengths of sides and the same measurements of angles. The triangle drawn on the hypotenuse of the right triangle has a side length, labeled (c). Let's try to find its area. The area of the triangle, if two sides and a confined angle are known, is equal to half the product of the lengths of two sides and the sine of the angle between them; that is, the area of the triangle drawn on the side (c) is half of the product of multiplying two adjacent sides that is (c) squared by the sine of the angle between them (pi) divided by 3.

Is this area equivalent to the area of the two triangles drawn on the sides of the right angle?

Let's find out the area of triangle drawn on the side (a). The area = half of the product of the lengths of two sides by the sine of the angle between them that is (pi) divided by 3 or 60 degrees.
 Also, the area of the triangle drawn on the side (b) is equal to half of the product of the lengths of two sides by the sine of the angle between them.

We note in this equation that there is one coefficient in all the limits and this coefficient can be omitted, by this coefficient is, 1/2 (Sine of pi over 3). If we omit this coefficient from the equation, we note that what remains is: $(c)2 = (a)2 + (b)2$, and this is stipulated in the Pythagorean Theory: the area of a triangle drawn on the hypotenuse of the right triangle, if the triangle is regular, equals the sum of the areas of the two triangles drawn on the sides of right angle.

Let's check if the theory is correct through this game.

Look! These are equilateral triangles drawn on a right-angled triangle. Let's see if the two-triangle area is equal to the area of the triangle drawn on the hypotenuse. Look . . . do you see that the result is correct? Namely, the area of any equilateral triangle drawn on the hypotenuse of the right triangle equals the sum of the area of the two triangles drawn on the sides of the right angle. Now what if the shapes drawn on the right-angled triangle are circles? We note that the diameter of the bigger circle is the hypotenuse of the right-angled triangle.

We want to find out the area of the larger circle to see if this area is

equal to the sum of the two circles drawn on the two sides of the right angle. You know that the area of a circle is $(r)^2(pi)$, where (r) is the radius, and that the diameter is (c), then the radius is $(c)/2$, so (r pi) is the area of the bigger circle.

 Is it equal to the sum of the areas of the two circles drawn on the two sides? The area of the circle diameter (b) is $(r)2(pi)$, and the area of a circle diameter (a) is $(r)2(pi)$, too.

Note in the three terms, there is a coefficient, $(1/2)^2(pi)$, by omitting this limit, i.e. $(1/2)^2(pi)$, the remainder of this equation is: the square of the hypotenuse = the sum of the squares of the two other hypotenuse, and that's what the Pythagorean Theory states: the area of the circle drawn on the diameter of the right-angled triangle is equal to the sum of the areas of the two circles drawn on the sides of the right angle.

Let's see if the hexagon satisfies the Pythagorean theory. You know that the hexagon is divided into six identical triangles, each with an area equal to the others, the side length of each is equal to the others, and the angles are 60 degrees, so then the area of a hexagon is equal to 6 multiplied by the area of one triangle.

Let's find the area of the polygon hexagon drawn on the hypotenuse, which is the side (c), so the area is 6 multiplied by the area of a triangle, which is the square of the side multiplied by the sine 180/3 i.e. 60 degrees.

Are the areas of the hexagons drawn on the sides of the right angle equivalent? Let's take the hexagon drawn on the side (a), its area is 6 multiplied by the area of triangle $(a)^2$sine of 60 degrees, also the area of the hexagon drawn on the side (b) is 6 of the area of triangles $1/2$ $(b)^2$sine.

We note in this equation that there is a coefficient that is repeated in the three limits, and by omitting this coefficient, 6 multiplied by 1/2 sine of

the angle, what remains is a square side of the right angle = the sum of the squares of the two sides of the right angle, and that's what the Pythagorean Theory states: the hexagon drawn on the hypotenuse of the right-angled triangle equals the sum of the areas of the two hexagons drawn on the sides of the right angle. Can we apply this theory if the set figures are pentagons or heptagons? The triangles in the pentagons are not regular. Let's think about this problem and I will see you later.

Welcome again! Thank you for your cooperation and achieving the amazing outcome. This is a regular pentagon; note that the triangle drawn within this figure is not regular, that is not equilateral, it is an isosceles , i.e. with two equal sides. The hypotenuse is not equivalent to any of the sides. To find out the area of this pentagon, we must find out the area of the triangle and multiply it by 5, because this figure is divided into five equal triangles.

Let's find out the area of the pentagon drawn on the hypotenuse (c), whose sides are equal, and each side (z) is: half of the product of multiplying the two sides by the sine of the angle between them: $2(\pi)/5$. So the area of the pentagon is 5 multiplied by the area of this triangle, however there is no right-angled triangle in this sentence, so we're trying to find out the value of (c) in terms of (z).
To make (c) the subject of the law, we can use the law of cosine: the square of the side of any triangle equals the sum of the square of the two other sides minus 2 multiplied by the first side by the second (cos) of the angle between them, that is (c)2 is equal to $(z)2 + (z)2 - 2 (z)2 (\cos)$ of the angle between them.

By producing the common factor, it becomes that (z) square 2 - 2 (cos)

This is the area of the pentagon drawn on the hypotenuse (c). Let's make the coefficient (a) equal to 5/2 × 2 - 2 (cos)2(pi)/5 sine 2(pi)/5.

If this is the coefficient (a), the area of the pentagon drawn on the hypotenuse (c) becomes $(ac)^2$. Using the same method, the area of the pentagon drawn on the side of the right angle (b) becomes $(ab)^2$, and the area of the figure drawn on the side (a) becomes $(aa)^2$. By omitting the coefficient (a), the output $(c)^2$ is equal to $(b)^2 + (a)^2$, as stipulated in the Pythagorean Theory.

Therefore, the area of the pentagon drawn on the hypotenuse of the right triangle is equal to the sum of the areas of the two pentagons drawn on the sides of the right angle.

Thank you for helping achieve this outstanding result! Now, I want to pose the following question: what if these figures are replaced with a prism, a cylinder, or a pyramid? Will the theory remain correct? Try to figure out a solution to this problem and I will see you shortly.

Hello again! Thank you for your cooperation. Let's see if the Pythagorean Theory applies to volumes. These three cubes are established on a right-angled triangle. This is the hypotenuse. These are the sides of the right angle. Note that the volume drawn on the sides of the two right angles is not equal to the volume drawn on the hypotenuse. That's surprising! But what if we amended these cubes and transformed each into a prism which has the same height . . . Let's see that.

This is a quartet prism, and these prisms have the same height with a square base. The side (c) is the hypotenuse of the right-angled triangle and the side (b) and the side (a) are the two sides of the right angle.

Let's try to find out the volume of each of these three prisms. The volume of the prism drawn on the hypotenuse (c) is the base area multiplied by the height, and as the base area is square, then the area is the side^2, i.e. the volume of the prism drawn on the side (c) is (hc) square by the base area by the height. Are the volumes of two prisms equivalent to the sides of the right angle? The prism drawn on the side (b) has the volume (hb)^2, times the area of the base by the height, and the volume of the prism drawn on the side (a) is the area of the base times the height (ha)^2.

By omitting the coefficient (h), we get $(c)^2 = (b)^2 + (a)^2$, and this is stipulated in the Pythagorean Theory: the volume of the prism drawn on the hypotenuse of the right triangle equals the sum of volumes of the two prisms drawn on the sides of the right angle, if they have the same height.

Let's see that. Look at these three prisms, their sides verify the Pythagorean Theory; this side is 10 cm and this one is 8 and this one is 6, therefore these three sides form a right triangle. Let's see if this volume fits these two volumes.

Let's check that together. See? The sum of the volumes of the two polygons drawn on the sides of the right angle has equaled the volume drawn on the hypotenuse of the right-angled triangle.

What if we replace the four-base prisms with five-base ones? Let's see if they fit the Pythagorean Theory or not. Let's check that.

This is a pentaprism, and this is another one. This prism has a hypotenuse of a right-angled triangle and these two prisms have the lengths of two other sides.
We want to know if these two volumes are equal to this volume . . . let's see that.

We have just learned that the area of the pentaprism drawn on the hypotenuse is equal to $(ac)^2$, and is equal to the area of the two prisms drawn on the two sides (a) and (b)
We know that the volume of the pentaprism is equal to the base area multiplied by the height, and since the height is the coefficient and for the three prisms, the height is (h), then the volume of the first prism becomes $(ahc)^2$, the volume of the second prism is $(aha)^2$, and the volume of the third prism is $(ahb)^2$. By omitting the coefficient (ah) from the three limits, the result becomes $(c)^2$ equal to $(a)^2 + (b)^2$, and this is stipulated in the Pythagorean Theory: the volume of the pentagon drawn on the hypotenuse of the right triangle is equal to the sum of the volumes of the two pentagons on both sides of the right angle.

Let's check that. This is a small polygon and this one is established on

the other side of the right.

Note that the volume of the two polygons has become equal to the volume of the polygon on the right-angled triangle, so we can apply the theory to pentagons and pentaprisms.

Let's verify the application of the Pythagorean Theory to the volume of a triangular prism. This triangular prism side length is 10 cm, and it's drawn on the hypotenuse of the right-angled triangle. This prism side is 6 cm, and this one's side is 8 cm. The three prisms are drawn on the side of a right-angled triangle. We have just learned that the area of the triangle is half the product of multiplying two sides by the sine of the angle between them, and we verified that the Pythagorean Theory applies to the area, and if we multiply the area by the same height (h), we note after omitting the coefficient (h) that the volume of the triangular prism drawn on the hypotenuse equals the sum of volumes of the two prisms drawn on the sides of the right angle. Let's verify that. Look . . .!

This prism is on the side of the right triangle, and this one is on the side of the second right angle. Note that we have been able to put the volumes of the two prisms drawn on the sides of the right angle on the volume of the prism drawn on the hypotenuse of the right angle; hence the theory is correct, and thus can be applied to all the polygons whose side lengths are (n) no matter what length the sides may be.

Verify that at home and now let's return to the problem of the juice seller. Let's think about how we can solve his problem now that we have studied these principles. I will see you later.

Hello again! Now that we are aware that the Pythagorean Theory applies

to volumes, is it possible to solve the problem of juice seller? Let's see if we can find out the diameter of this circle, and this one; the hypotenuse is the volume of the cylinder by which we can unload the volumes of these two cylinders. Let's see how to find out the diameter of this cylinder. We fix the stick here in this position and the largest effect it leaves will be the diameter of this circle because the diameter is the hypotenuse of the circle; then this is the diameter of the small circle, this is the diameter.

We can find out the diameter of the other circle in the same way. The longest hypotenuse is the diameter.
This is the diameter. Now, since the volumes apply to the Pythagorean Theory, then these two diameters form the two sides of the right angle, and the diameter of the volume including these two volumes, will be the hypotenuse. Let's try to find out the diameter of the circle into which we need to unload the cylinder, trying to find the hypotenuse of this triangle.

Note this is the hypotenuse of the right triangle, it is the same as the diameter of the big cylinder whose volume equals the volumes of these two cylinders. Let's search for it between the cylinders; we have not used any measurement, just a stick. Note that the diameter is larger than the diameter of this cylinder because it is outside the circle, and here also it is larger because it is outside the circle. Note here is the same diameter, then this is the appropriate cylinder, this is the appropriate cylinder. Let's see that.

See? We do not need any measure or too much effort, and we were able to unload the two containers into one container, but in the beginning of the lesson we could not find out the required polygon; the sand was in the container that did not fit it.

Now, can we apply that to other figures? Let's see that without measurements. Some housewives are having trouble, wanting to pour two containers into one because the space they have is narrow. How can we help them? We believe that Pythagoras has helped because the two volumes are equal to the volume of the third one, if we apply the Pythagorean Theory to them. Let's check that.

Let's find out the side of the hexa-prism…here it is! And let's find the side of the other hexa-prism…here it is, too! We can find out the third prism, whose volume is equivalent to the volumes of the two prisms if it has the hypotenuse side of the right-angled triangle. Let's find that out. This is the side of the prism, which we are looking for. Housewives can take this stick and search for a prism with a side equal to the stick. Should they search for it, they will find out that this prism is applicable to it. Notice that it has the same side. Let's see that.

Thus, we can help the housewife pouring two containers into one, through the application of the third volume established on the hypotenuse of the triangle of the circle. But think about a different application of the Pythagorean Theory and work together. I hope you have enjoyed this lesson as much as I enjoyed preparing it. Thank you and goodbye in your other lessons. Thank you.

Hello. I'm Ghada Marmash. Thank you for choosing this lesson. I hope

you have enjoyed it as much as I enjoyed preparing and teaching it. We must now offer you the prerequisites for this lesson. Students should know the following:

- Pythagorean Theory.
- The area of a triangle if two sides and a sine are known.
- Law of cosine.
- The area of the circle.
- The volumes of different polygons if the base is regular.

Concerning the intervals, in the first one, the students unload the sand from a polygon to another and record their observations, so that they may reach a conclusion. In the second interval, the students draw various figures on the sides of a right-angled triangle such as triangles, hexagons, pentagons, heptagons, and the teacher helps them find out the different areas and guides them to the law of sine and cosine. In the third interval, students draw regular pentagons on the sides of the right-angled triangle, and it is surprising that the theory applies to these figures although the triangles which the figure is divided into are not regular, and the teacher guides them to the law of cosine and helps them find the base.

In the fourth interval, each group finds out the volume of the polygon with a known base area by multiplying the area by the height. Each group should prove that the volume established on the hypotenuse equals the sum of the two volumes. One of the two groups should take a tri-gone and another one should take a quartet one and another a pentaprism, and so on, until we get to the result, so we can prove the theory.

In the fifth interval, the students try to help the juice seller after they have learned that the Pythagorean Theory applies to volumes. Using a ruler or stick to find the side length of the required polygon, they take the diameter of the cylinder using the ruler and then find out the diameter of the other cylinder, and form a right-angled triangle. The hypotenuse is the volume of the required cylinder. Then, they try to help a builder and a housewife using a ruler, and try to form a right-angled triangle and find out the appropriate volume, and thus they have applied what they have learned in this lesson to find out the appropriate volume for the other two volumes. I hope you have enjoyed this lesson and thank you for your help and attention. Thank you.