MIT BLOSSOMS INITIATIVE

"The Broken Stick Problem"
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Segment 1

Hi! My name is Dick Larson and I'm a teacher at MIT here in Cambridge, Massachusetts, USA. I hope you're feeling fine today and full of energy. We have an interesting challenge problem for you that's going to build on the math skills that your teacher and you have been working on these past few weeks and maybe months.

Today's problem deals with triangles and if you're currently in a geometry class in a high school, that's sometimes called the science of triangles, and so it's not inappropriate for us to study triangles, but we be doing it in a different way. And this problem is fun, engaging, it's going to require your participation and you and your teacher are going to help us solve and work out this problem.

OK. So here is the problem. We have here a yardstick and it's numbered one to thirty six inches. These are in so-called British distance units. Many of you may also talk about meters and centimeters and those kinds of units. That's fine too. Basically we're going to look at this yardstick and do something which is kind of interesting. Now you might say why didn't I use a meter stick? Well, locally here in Cambridge it didn't cost me very much to buy one of these wooden yardsticks, but a meter stick would have cost me like a hundred times more. So that's why I chose a hard stick. Where you are, if you don't have access to either one, you could even use a stick from a tree, if you can mark it off in equal length segments.

So part one here. We want to select two random points on this yardstick. By that I mean we want to have a point here and a point there, or wherever it is, random points that before the fact, any one of the numbers, let's say one to thirty six inches is equally likely. So we want to generate two random numbers, equally likely to be anything along the length of this yardstick. So I want you in class with your teacher and with your friends next to you to discuss how you might generate such random points. Why don't you do that for the next two or three minutes and we'll come back and advance this problem a little bit further? See you in a couple of minutes.

Segment 2

Hi! And welcome back. Now we're going to select two random numbers, and I'm sure you've discussed three, four, five different ways of getting random numbers. We're going to do it the easy way here. I put 36 random numbers in this box right here and they're all mixed up and we're going to select two at random. They're numbered one to 36. Let me pick out the first one here. [He reaches in. He pulls out] Ah, it is a 24. OK? So on this yardstick here we're going to number 24 right there. And now we're going to go and select the second random number. Let's see if I can find one in here. They're kind of at the bottom. OK. Here's one. Ah! 10. See, there's 10. And the rest of them are in there, you see them? They're all in there. OK? So let us put a 10 over here like this. There we go. So we first of all had 24 and then we had 10.

So now I can define the problem in terms of triangles. We said this is a study in the science of triangles. Here's the question. Suppose I were to do this 10,000 times with 10,000 different yardsticks like this and different sets of random numbers each time, pull out the random numbers independently each time. And suppose, as a thought experiment, I would actually cut this stick at the two points where the chalk is. Right here and right here, OK? How many times out of the 10,000 do you think I could form a triangle with the three pieces? How many times could I form a triangle? I want you to discuss this with your friends next to you in class, with your teacher, and maybe somebody will even volunteer your names and your teacher can put your name on the blackboard with your estimate as to how many times out of 10,000 you think we could make a triangle with the three pieces. Would it be 5,000 times? 9,000 times? 10,000 times? Zero? OK? Think about it. Put your name on the blackboard with a guess and then we'll come back and solve the problem. See you soon. Bye!

Segment 3

Welcome back! I hope you've had a nice discussion about this now and each of you have made some estimates about if we cut this thing 10,000 times in 10,000 different experiments, 10,000 different sticks, how many times we could form a triangle. Well guess what? We're going to do this live experiment here today because I have this device that can actually cut this yardstick into three pieces at the points that we marked here, OK? So let's do that. I have to be very careful that this does not slide or cause any injury. So we're going to cut through here first at 24. There we go at 24. And then I have to cut through at 10. This saw is very sharp so I have to be careful. There we go! All right. Ladies and gentlemen, we have the three pieces so obtained and let's ask the question, "Can we form a triangle with these three pieces?" Well look. We do like this and we do like this. Ta Da! There's a beautiful, wonderful triangle. Not too far from an equilateral triangle just by chance here with these random numbers.

Does this mean we can always form a triangle with these three pieces? No. No. Before the cameras went on, in a rehearsal in this room, I did this experiment with another set of random numbers pulled out of here and I came up with these three pieces. All right? These three pieces are just as good as the three pieces we just had. Can I form a triangle here? Let's see. OK. We try to attach this one at the end and we try to stretch this to make a triangle. Ohhh it doesn't work. Can't do it! Can't do it. Why? Because this long piece is too long. OK? When one of the pieces is too long, we can't form a triangle.

So there we have the situation. Sometimes we can form a triangle, sometimes we can't. Let's solve this problem together and figure out exactly what is the chance that we could form a triangle if we do this experiment. And you could do it live in your own home or in your classroom as well. Now what we're going to do to solve this is do it in a systematic manner. And what I say is we need to follow four steps to happiness. So these four steps to happiness are:

1) Define the variables of interest.
2) Draw the space in which they take on values
3) Define an event in that space and determine its likelihood
4) And finally, identify the event of interest, in this case being able to form a triangle, and try to solve the problem.

OK. So we are going to go over here and define the variables first of all. So let's look at step # 1. Define the variables. Step # 1. Well, we have a yardstick, so let's draw this yardstick like this. That's supposed to be a straight line. Zero to one yard. See we can use different dimensions. We could use one to 36 inches, or we could use zero to one yard. Let's stick with the yards because it makes it easier. Easier to deal with zero to one, rather than zero to 36. OK. So here's the zero, here's the one. Now define the variables. If you recall, the first random number we picked out was 24, right here. We'll call that X1. That's the first one we draw. And the second one we took was 10 and that's an experimental value for X2. OK? So in the four steps to happiness we've already now done step one. We defined the variables.

Now we draw the space in which they take on values. So what do I mean by that? Well, this is what I mean by that. Here's values for X! on the horizontal axis. Here's values for X2 on the vertical axis. Now I'm doing this by eyeballing it, so I don't know exactly how this is going to come out, but it's supposed to be a square. And the dimensions zero to one here and zero to one here. And any particular outcome of this experiment is a point right in here. Let's look. If this was 24 and 10, OK? So this is about two thirds. So X1 was this value and X2, ten is a little bit less than a third over there. OK? So the outcome of our experiment is shown by that point right there. That's the outcome of our experiment when we pulled random numbers from that basket. OK. So any particular time we run this experiment, the outcome of the experiment is shown by a numerical value for X1, a numerical value for X2. Both are between zero and one, so they are characterized by a point in this space. So we have now drawn the space in which X1 and X2 take on values. So we have done step one. We have done step two.

Now define an event and determine its likelihood. Well, let's erase what we've drawn up here. We know that any point in this space corresponds to a possible outcome. Suppose we have an event like this. Let's see. X1 greater than three quarters, AND X2 between one quarter and three quarters. OK? Suppose we have an event like this. So three quarters would be something like right here. Let's call that three quarters. And X2 between one quarter and three quarters. Maybe that's a quarter there and maybe this is three quarters here. So we want both this to occur for X2, this to occur for X1. So the event of interest here is that little rectangle right there. So anytime we get an outcome of the experiment in this rectangle, this event has occurred. Any time we get a point outside of that rectangle, which is far more likely, this event has not occurred. OK?

Define an event and determine its likelihood. So we've now defined an event, which is drawn in this space. We draw it in this space. Now determine its likelihood. Well, how do we do that? We don't really know the theory of probability but we don't need to because look. Any point in this square is equally likely. OK? So then the likelihood that we end up in this rectangle versus some place else, is proportional to the area of the rectangle. If the rectangle were twice as big as it is now, maybe over to here, we'd be twice as likely to show up in that rectangle because we're kind of uniformly distributed over this whole thing. And since the area of the entire big square is one, the likelihood or the probability that we end up here is just equal to its area. So the probability of this event equals one quarter this way, times a half this way, equals one eighth. That is the probability of that event. So now we know how to characterize the events and determine their likelihood. Good! We're three quarters of the way through the four steps to happiness. I hope you're feeling very, very happy right now because we're almost done there.

So what do we have to do now? Why have we done this? Because we now need to identify the event of interest and solve the problem. Now the event of interest is of the three pieces I obtained by cutting this yardstick, I can form a triangle. I want you to think about this and talk about it to your neighbors and with your teacher and see if you can formalize the situation of why something like this you can form a triangle, and something like this you can't. We know that in some sense no one piece can be too big. Think about it. Talk about it. See you back here in a few minutes.

Segment 4

Welcome back. By now I'm sure you've had a very active discussion in your class about what conditions need to prevail in order for a triangle to be formed. You notice that this one here we can't form a triangle because this piece on the bottom is too long. Now if you formalize this in your class, you see what the definition of too long is. This one is more than half a yard length long. If one of the three pieces is more than half a yard length, then you can't stretch the other two to make a triangle. But if each of the three pieces, EACH of the three pieces is less than a half a yard length, then you can make a triangle. And I'm sure that was the outcome of your class discussion. So now let's put that onto the blackboard and look at step # 4 here. Identify the event of interest. And basically what we have to do now in concluding this four steps to happiness, to be really. really happy is we have to put a picture on this space and inside this picture any point means you can form a triangle. Outside the picture any point means you can't form a triangle. So now we're looking at the event of interest being able to form a triangle. Maybe we put a big triangle there to indicate this.

OK. So let's look at it. Here we have X1 and X2. Let's just… it's a little bit complicated because X1 could be over here to the left of X2. So there's no order in here. X1 could be greater than X2, it could be less than X2. So let's do some conditioning and let's look at the situation the way we found it. So if you go this side here, of this 45 degree line, if you go to this side, you find out that X1 is greater than X2, which is the situation that we found when we did our experiment. So X1 is greater than X2. Just as we have this drawing right here. So what we need to do is write an equation for each of the three pieces. So the length of piece number one is just right here, X2. OK? Piece number two. The length of piece #2 is X1 minus X2. Got that? OK? Now we have one more piece to go here. And that is this one right here, the length of this piece is one minus X1. One minus X1 is piece #3. All right.

Now, we said that for a triangle to be formed, no piece can be too long. And what we mean by this is no piece can be greater than a half a yard. So we have constraints on each of the three pieces. Each of the three pieces must be less than half a yard length long. How did we do this? So this must be less than a half. This must be less than a half. That must be less than a half. We need to find in our space the area, the diagram, the picture, which corresponds to each of these three constraints being valid. OK. So X2 must be less than a half. X2 is over here. Here's a half. Anything below this line is OK. So X2 must be less than a half. All right? So we need that. Let's' go to the third piece here. The third piece here we could put X1 on the right hand side. Take the half, subtract both sides by one half there, so that means that X1 must be greater than a half. OK? So here's a half right there and X-1X1 must be greater than a half, so we must be to the right of this line for the constraint on the piece #3 to be valid. Now we have this complicated one right here. So let's look at this. Let's bring the X2 to the right hand side. Bring the half over here. X1 minus one half and we need X2 greater than X1 minus a half. OK? So if we were to write an equation, X2 = X1 minus one half. I want you to think about this. But that equation, when we're in this square would look like this. It would have a slope of one, X2 goes up as X1 does, but it's offset by minus one half. So if you do that, you find you get a line like this, goes down here to minus one half. And we have to be… let's see… X2 has to be greater than this line, greater than that line, so we have to be up here like this. OK? So we have three constraints. We have to look for the intersection of those three constraints, the set of points where each of the three constraints works. What is that? That is this triangle right here! All right? That is part of the picture. That is part of the solution. But it's only part of the solution because it characterizes it for the case X1 greater than X2.

Now we're going to take a final pause here and we want you to finish this with your classmates and with your teacher and finish this diagram and be able to answer the question if we did this 10,000 times, about how many times do you think we could form a triangle with the three pieces we cut? So go to it. Finish the problem. We'll be back for one more minute at the end. See you soon!

Segment 5

Congratulations! You and your teacher and your class have discovered that the probability that you could form a triangle from this experiment is 25% because in class you've just done the symmetric reverse of this. When we have X1 less than X2 and you found out the other part of this looks like this. And the area of these two triangles summed together, it's one eighth plus one eighth, is a quarter and that's the answer. So if you did this roughly 10,000 times, approximately—don't know exactly because this is uncertain in probability—2,500 times you could form a triangle. Now I don't know how you did with your guesses but most people guessed much higher numbers. Two thirds of the time. Three quarters. Usually there are people who say, "Well if you have three pieces you could form a triangle 100% of the time." So I don't know how you did. It's very, very rare though for somebody to estimate as low as 25% which is the correct answer. And that shows you the benefits of thinking systematically and carefully and step-by-step. In this case going through the four steps to happiness to solve our problem like this.

One of the side benefits is we've introduced you to probability without really emphasizing probability. And those of you who go on to study probability, which is analysis of problems operating in uncertain environments, you'll find that these four steps to happiness are critical to solving all kinds of interesting and fascinating probability questions. But that's all for today. I want to thank you for your time, your energy, your commitment. And I hope to see you again soon! So long!

Teacher Guide Segment

Hi! This is Dick Larson again and now I'm talking to you the teachers, the high school teachers, probably in a math class who will be wondering what should I do during these four breaks that we have here in this blended learning module on broken sticks. So let me just go through it. OK? And you have a lot of flexibility yourself as to how you want to manage these breaks. The idea is to get the students engaged, get them committed, get them excited about solving this problem using fundamental principles, back to basics.

So the first break is basically how do you generate random numbers. Well in the MIT class that we teach here, and I do this every year. I do the same experiment live before the students, is basically I say, "Well all of those who are wearing wrist watches raise your hand." Of course I don't wear a wrist watch but many of them do. And then I say, "For all of those with your hands up, if you can read the seconds in addition to minutes and hours, keep your hands up." OK. "For those of you with your hands still up, for those who have not reset your watch in the last three or four weeks, keep your hands up." OK? Usually by this time in class of about 30 I have six or eight students who still have their hands up. And then I say, "OK. To those students with your hands up, on the count of three I want you to write down the seconds reading from your watch. One, two three. Boom." And they all write down dutifully the seconds reading from their wristwatch. And then I'll pick two of them at random from the class, maybe John over here and Jane over there. And I'll say, "Well what number did you get?" And maybe we got two numbers from the wrist watches. Maybe 44 and 15. And the problem with the yardstick is that these numbers are random numbers between zero and 60 and I need to have random numbers between zero and 36. So what do we do? For X1 we divide by 60. We scale this and multiply by 36. And that's for X1. For X2 we do the same thing. We divide by 60 and multiply by 36. And if I did the arithmetic right, this is 26.4 and this is 9. So this scaling from wrist watch readings gets me the random numbers that I'm looking for. So that's one way of doing it.

But many students don't wear wrist watches. Many teachers don't wear wrist watches. So that's not really a fail safe way to do it. Some classes actually create like a wheel, a spinning wheel you could spin around and have it marked from one to 36. If you're using a meter and you have it separated into centimeters, you could have a wheel labeled one to 100. That's fine too. Or you could take a coin and paint on both sides of it a narrow strip line and spin the coin in the air and when it falls, the direction it's headed in is a direction that maybe you have the directions numbered one to 36 uniformly spaced. There are different ways to create random numbers, but the key thing is to remember that they have to be uniformly distributed over the length of the yardstick. OK?

Now some students might say, "Well gee isn't this a continuous problem in the sense that X1 can take on any value between zero and 36 or between zero and one yard. It doesn't have to take an integer value." And that's true. So if the students are disturbed by the fact that we only put 36 numbers into this box here, you could say, "Well let's make it more accurate. So we'll start with one quarter inch, one half an inch, three quarters, one inch, one and a quarter, one and a half etc." So they could put four times as many numbers in there. For demonstration purposes, treating this as an integer problem is fine. And the way we solve it on the blackboard, we solve it exactly as if it's a continuous problem. So some advanced students may ask about the integer approximation. Say, "Well, that's an approximation. This happens in engineering and science all the time. You want to make it more accurate? Put in more numbers and divide the zero to 36 inches into smaller segments." OK? So that's fine there. That's about all I'm going to say about segment number one, which is just basically to engage the students and have the interactivity with the class.

OK. Segment number two is basically a fun segment. The idea is to get the students to express to themselves and to you and to their classmates their intuition. Is it likely to form a triangle 80% of the time? 60% of the time? 100% of the time? And to get them to commit to it by putting their name—you can put their name on the blackboard next to their estimate. OK? So usually when I do this at MIT I get six or seven names with six or seven estimates and the estimates typically vary from 100% maybe down to 50 or 40%. Rarely do I get anyone who estimates as low as the correct answer is, which is 25%. But that's a fun thing. And usually by the end of segment two the students are really committed, really invested and really want to solve this problem. They're really engaged in it.

OK. For pause #3 the idea is that they need to find the event. They need to think about what conditions do we need to have solved so that a triangle can be formed with the three pieces. And so we have to formalize an intuition. And this is going back to fundamentals. So they think of the intuition. They now realize that sometimes we can form a triangle, sometimes we can't. We can't if one of the pieces is too long and it can't just be the center piece, or it can't be the left most piece or the right most piece. Any one of the three pieces, each of them has to be not too long. And then the idea is how do you crystallize, how do you formalize mathematically not too long? And then they think about it and all of a sudden usually there's an ah-ha moment. Not too long means a half, a half a yard. If each of the three pieces is less than a half a yard, boom! We can make a triangle. It may not be a pretty triangle, but we can make a triangle because the smaller ones will sum to a greater length than the longest one and we can form a triangle. So that's the idea is to get them very much involved with that. Maybe they haven't formalized it yet the way we do on the video, following that segment. Some of them may go so advanced and so far that they actually cover what I cover in that next video segment. And that's fine too. Maybe they can solve the problem right there. That's great!

OK. In pause #4, the last pause, is basically I've shown the lower triangle in the square of outcomes, X1, X2 and that's for the case when X1 is greater than X2. So we have this lower triangle here. And what I'm asking you to do in class is to have the students figure out this upper triangle here. And they can either do it by going through the same three steps that I did here, this is the sure fire way of doing it. Identify the length of piece one, identify the length of piece two and piece three. Then you can construct this the same way I constructed the first one. OR, they can invoke a very powerful idea from mathematics and that is symmetry. And if they invoke the symmetry argument, they can almost draw this by inspection. OK?

So that's basically the summary. And you'll see that I've given these notes also in the teachers' guide that accompanies this interactive video. And if you really want to see this whole thing worked out in color animation on the worldwide web, let me point to the fact that there's a URL which is right here, it's a long one. I don't expect you to memorize this here. But this URL will also be in your teachers guide. And so those of you who want to see basically a space ship come in from outer space with laser beams and break a stick at random points and then solve this in color animation mode, that's available on the web. So thank you for trying this module. There's a website that you can make comments, or send me an e-mail directly at rclarson@mit.edu. And we look forward to your comments and constructive feedback. Thank you very much.

END OF TRANSCRIPT