Rational numbers vs. Irrational numbers

by

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in cooperation with

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An MIT BLOSSOMS Module
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“The ultimate Nature of Reality is Numbers”

A quote from Pythagoras (570-495 BC)
“Wherever there is number, there is beauty”
A quote from Proclus (412-485 AD)
Traditional Clock plus Circumference

1 min = $\frac{1}{60}$ of 1 hour

Circumference length is $\pi$

Diameter 1

Rational numbers vs. Irrational numbers
An Electronic Clock plus a Calendar

15:02:09
19/09/2005

Hour : Minutes : Seconds
dd/mm/yyyy

1 month = \frac{1}{12} \text{ of 1 year}

1 day = \frac{1}{365} \text{ of 1 year (normally)}

1 hour = \frac{1}{24} \text{ of 1 day}

1 min = \frac{1}{60} \text{ of 1 hour}

1 sec = \frac{1}{60} \text{ of 1 min}

Rational numbers vs. Irrational numbers
TSquares: Use of Pythagoras Theorem
Golden number $\varphi$ and Golden rectangle

Roots of $x^2 - x - 1 = 0$ are $\varphi = \frac{1 + \sqrt{5}}{2}$ and $-\frac{1}{\varphi} = \frac{1 - \sqrt{5}}{2}$
Golden number $\varphi$ and Inner Golden spiral

Drawn with up to 10 golden rectangles
Outer Golden spiral and L. Fibonacci (1175-1250) sequence

\[ F = \{1, 1, 2, 3, 5, 8, 13, \ldots, f_n, \ldots\} : f_n = f_{n-1} + f_{n-2}, \ n \geq 3 \]

\[ f_n = \frac{1}{\sqrt{5}} \left( \varphi^n + (-1)^{n-1} \frac{1}{\varphi^n} \right) \]
Euler’s Number \( e \)

\[
\begin{align*}
\text{s}_3 &= 1 + \frac{1}{1!} + \frac{1}{2} + \frac{1}{3!} = 2.6666\ldots66\ldots, \\
\text{s}_4 &= 1 + \frac{1}{2} + \frac{1}{3!} + \frac{1}{4!} = 2.70833333\ldots333\ldots, \\
\text{s}_5 &= 1 + \frac{1}{2} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} = 2.7166666666\ldots66\ldots.
\end{align*}
\]

\[
\lim_{n \to \infty} \left\{ 1 + \frac{1}{2} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \ldots + \frac{1}{n!} \right\} = e = 2.718281828459\ldots\ldots
\]

\( e \) is an irrational number discovered by L. Euler (1707-1783), a limit of a sequence of rational numbers.
Definition of Rational and Irrational numbers

- A **Rational number** $r$ is defined as:

$$ r = \frac{m}{n} $$

where $m$ and $n$ are integers with $n \neq 0$.

- Otherwise, if a number cannot be put in the form of a ratio of 2 integers, it is said to be an **Irrational number**.
Distinguishing between rational and irrational numbers

Any number $x$, (rational or irrational) can be written as:

$$x = I + f$$
Distinguishing between rational and irrational numbers

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- $I$ is its integral part;
Distinguishing between rational and irrational numbers

Any number \( x \), (rational or irrational) can be written as:

\[ x = I + f \]

- \( I \) is its integral part;
- \( 0 \leq f < 1 \) is its fractional part.
Examples

• \(\frac{48}{25} = 1 + 0.92\)

• \(\frac{8}{3} = \)

• \(\frac{17}{7} = \)

• \(\sqrt{2} = \)

• \(\pi = \)

• \(\varphi = \frac{1 + \sqrt{5}}{2} = \)

Rational numbers vs. Irrational numbers
Answers to Examples

- \( \frac{48}{25} = 1 + 0.92 \)
- \( \frac{8}{3} = 2 + 0.66666666..... \)
- \( \frac{17}{7} = 2 + 0.4285714285714..... \)
- \( \sqrt{2} = 1 + 0.4142135623731..... \)
- \( \pi = 3 + 0.14159265358979..... \)
- \( \varphi = 1 + 0.6180339887499...... \)
Distinguishing between rational and irrational numbers
Distinguishing between rational and irrational numbers

1. As \( x = I + f, \) \( I: \) Integer; \( 0 < f < 1: \) Fractional.
Distinguishing between rational and irrational numbers

1. As $x = I + f$, $I$: Integer; $0 < f < 1$: Fractional.

2. → Distinction between rational and irrational can be restricted to fraction numbers $f$ between $0 < f < 1$. 

Rational numbers vs. Irrational numbers
Position of the Problem

\[ R = \{ \text{Rational Numbers } f, \ 0 < f < 1 \} \]
\[ I = \{ \text{Irrational Numbers } f, \ 0 < f < 1 \} \]

The segment following segment \( S \) represents all numbers between 0 and 1:

\[ S = R \cup I \text{ with } R \cap I = \emptyset \text{ empty set.} \]

- **Basic Question:**
Position of the Problem

\[ \mathcal{R} = \{ \text{Rational Numbers } f, 0 < f < 1 \} \]

\[ \mathcal{I} = \{ \text{Irrational Numbers } f, 0 < f < 1 \} \]

The segment following segment \( S \) represents all numbers between 0 and 1:

\[ S = \mathcal{R} \cup \mathcal{I} \text{ with } \mathcal{R} \cap \mathcal{I} = \emptyset \text{ empty set.} \]

- **Basic Question:**
- If we pick a number \( f \) at random between 0 and 1, what is the probability that this number be rational: \( f \in \mathcal{R} \)?
The Decimal Representation of a number

Any number $f : 0 < f < 1$ has the following decimal representation:

\[
\text{Notation} \quad f \quad \overset{\equiv}{=} \quad 0.d_1d_2d_3\ldots d_k\ldots
\]

\[d_i \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}\]

\[f = d_1\left(\frac{1}{10}\right) + d_2\left(\frac{1}{100}\right) + d_3\left(\frac{1}{1000}\right) + \ldots + d_k\left(\frac{1}{10^k}\right) + \ldots\]

with at least one of the $d_i$’s $\neq 0$. 

Rational numbers vs. Irrational numbers
Main Theorem about Rational Numbers

The number $0 < f < 1$ is rational, that is

$$f = \frac{m}{n}, \ m < n,$$

if and only if

its decimal representation:

$$f = 0.d_1d_2d_3\ldots d_k\ldots$$

$$= d_1\left(\frac{1}{10}\right) + d_2\left(\frac{1}{10^2}\right) + d_3\left(\frac{1}{10^3}\right) + \ldots + d_k\left(\frac{1}{10^k}\right) + \ldots$$

takes one of the following forms:
Main Theorem about Rational Numbers

The number $0 < f < 1$ is rational, that is

$$f = \frac{m}{n}, \ m < n,$$

if and only if

its decimal representation:

$$f = 0.d_1d_2d_3...d_k...$$

$$= d_1\left(\frac{1}{10}\right) + d_2\left(\frac{1}{10^2}\right) + d_3\left(\frac{1}{10^3}\right) + ... + d_k\left(\frac{1}{10^k}\right) + ...$$

takes one of the following forms:

f is either **Terminating**: $d_i = 0$ for $i > l \geq 1$
Main Theorem about Rational Numbers

The number $0 < f < 1$ is rational, that is $f = \frac{m}{n}$, $m < n$, if and only if its decimal representation:

$f = 0.d_1d_2d_3...d_k...$

$= d_1\left(\frac{1}{10}\right) + d_2\left(\frac{1}{10^2}\right) + d_3\left(\frac{1}{10^3}\right) + ... + d_k\left(\frac{1}{10^k}\right) + ...$

takes one of the following forms:

- $f$ is either **Terminating**: $d_i = 0$ for $i > l \geq 1$
- or $f$ is **Non-Terminating** with a repeating pattern.
Proof of the Main Theorem about Rational Numbers

**Theorem**

The number $0 < f < 1$ is rational, that is $f = \frac{m}{n}$, $m < n$, if and only if its decimal representation:

$$f = 0.d_1d_2d_3...d_k...$$

is either **Terminating** ($d_i = 0$ for $i > l \geq 1$) or is **Non-Terminating** with a repeating pattern.
Proof of the only if part of Main Theorem about Rational Numbers

Proof.
Proof of the only if part of Main Theorem about Rational Numbers

Proof.

1. If $f$ has a terminating decimal representation, then $f$ is rational.
Proof of the only if part of Main Theorem about Rational Numbers

Proof.

1. If $f$ has a terminating decimal representation, then $f$ is rational.

2. If $f$ has a non-terminating decimal representation with a repeating pattern, then $f$ is rational.
Proof of the first Statement of only if part

**Statement 1:** If \( f \) has a terminating decimal representation, then \( f \) is rational.  
**Consider:**

\[
f = d_1 \left( \frac{1}{10} \right) + d_2 \left( \frac{1}{100} \right) + d_3 \left( \frac{1}{1000} \right) + \ldots + d_k \left( \frac{1}{10^k} \right)
\]

then:

\[
10^k f = d_1 10^{k-1} + d_2 10^{k-2} + \ldots + d_k.
\]

implying:

\[
f = \frac{m}{10^k} \quad \text{with} \quad m = d_1 10^{k-1} + d_2 10^{k-2} + \ldots + d_k
\]
Example
Example

\[ 0.625 = \frac{625}{1,000} = \frac{125 \times 5}{125 \times 8} \]
Example

\[0.625 = \frac{625}{1,000} = \frac{125 \times 5}{125 \times 8}\]

\[0.625 = \text{after simplification: } \frac{5}{8}\]
Proof of the second Statement of only if part

Statement 2: If \( f \) has a non terminating decimal representation with repeating pattern, then \( f \) is rational. Without loss of generality, consider:

\[
f = 0.d_1 d_2 d_3 \ldots d_k = 0.d_1 d_2 d_3 \ldots d_k d_1 d_2 d_3 \ldots d_k d_1 d_2 d_3 \ldots d_k \ldots
\]

\[
f = d_1 \left( \frac{1}{10} \right) + d_2 \left( \frac{1}{100} \right) + d_3 \left( \frac{1}{1000} \right) + \ldots + d_k \left( \frac{1}{10^k} \right) + \frac{1}{10^k} \left[ d_1 \left( \frac{1}{10} \right) + d_2 \left( \frac{1}{100} \right) + d_3 \left( \frac{1}{1000} \right) + \ldots + d_k \left( \frac{1}{10^k} \right) \right] + \frac{1}{10^{2k}} [..].
\]

then:

\[
10^k f = d_1 10^{k-1} + d_2 10^{k-2} + \ldots + d_k + f.
\]

implying:

\[
(10^k - 1) f = m \iff f = \frac{m}{n}
\]

Rational numbers vs. Irrational numbers
Example on Proof of the second Statement

\[
f = 0.\overline{428571} = 0.428571428571428571... \]

\[
f = 4 \left( \frac{1}{10} \right) + 2 \left( \frac{1}{100} \right) + 8 \left( \frac{1}{10^3} \right) + 5 \left( \frac{1}{10^4} \right) + 7 \left( \frac{1}{10^5} \right) + 1 \frac{1}{10^6} + \frac{1}{10^6} (f) \]

\[
10^6 \times f = 4 \times 10^5 + 2 \times 10^4 + 8 \times 10^3 + 5 \times 10^2 + 7 \times 10 + 1 + f \]

\[
(10^6 - 1) \times f = 428,571 \]

\[
f = \frac{428,571}{10^6 - 1} = \frac{428,571}{999,999} \]

After simplification:

\[
f = \frac{428,571}{999,999} = \frac{3 \times 142,857}{7 \times 142,857} = \frac{3}{7} \]
Proof of the “IF PART”

\[ f = 0.d_1d_2d_3...d_k... \in \mathcal{R} \]

\[ \Downarrow \]

\begin{itemize}
  \item \( f \) has a terminating representation,
  \item or
  \item \( f \) has a non-terminating representation with a repeating pattern.
\end{itemize}
Tools for Proof of the if part of Main Theorem about Rational Numbers

Two tools to prove this result:
Tools for Proof of the if part of Main Theorem about Rational Numbers

Two tools to prove this result:

1. Euclidean Division Theorem
Tools for Proof of the if part of Main Theorem about Rational Numbers

Two tools to prove this result:

1. Euclidean Division Theorem
2. Pigeon Hole Principle
First Tool: Euclidean Division Theorem

\( M \geq 0 \) and \( N \geq 1 \).

Then, there exists a unique pair of integers \((d, r)\), such that:

\[
M = d \times N + r,
\]
or equivalently:

\[
\frac{M}{N} = d + \frac{r}{N}
\]

\( d \geq 0 \) is the quotient of the division, and \( r \in \{0, 1, \ldots, N - 1\} \) is the remainder.
Application of Euclidean Division Theorem on $f$, $0 < f < 1$

\[ f = \frac{m}{n} = d_1 \left( \frac{1}{10} \right) + d_2 \left( \frac{1}{100} \right) + d_3 \left( \frac{1}{1000} \right) + \ldots + d_k \left( \frac{1}{10^k} \right) + \ldots \]

\[ \frac{10m}{n} = d_1 + f_1 \text{ where } f_1 = d_2 \left( \frac{1}{10} \right) + d_3 \left( \frac{1}{100} \right) + \ldots + d_k \left( \frac{1}{10^{k-1}} \right) + \ldots \]

<table>
<thead>
<tr>
<th>Equation</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$10m = d_1 n + r_1$</td>
<td>$\frac{10m}{n} = d_1 + f_1$</td>
</tr>
<tr>
<td>$10r_1 = d_2 n + r_2$</td>
<td>$f_1 = \frac{r_1}{n} = d_2 \left( \frac{1}{10} \right) + \ldots$</td>
</tr>
<tr>
<td>$10r_{k-1} = d_k n + r_k$</td>
<td>$\frac{10r_{k-1}}{n} = d_k + f_k$</td>
</tr>
<tr>
<td>$f_k = \frac{r_k}{n} = d_{k+1} \left( \frac{1}{10} \right) + \ldots$</td>
<td></td>
</tr>
</tbody>
</table>

Each of $r_1, r_2, \ldots, r_k, \ldots \in \{0, 1, \ldots, n - 1\}$
Application of Euclidean Division Theorem on \( f, \ 0 < f < 1 \)

\[
f = \frac{m}{n} = d_1 \left( \frac{1}{10} \right) + d_2 \left( \frac{1}{100} \right) + d_3 \left( \frac{1}{1000} \right) + \ldots + d_k \left( \frac{1}{10^k} \right) + \ldots
\]

\[
\frac{10m}{n} = d_1 + f_1 \text{ where } f_1 = d_2 \left( \frac{1}{10} \right) + d_3 \left( \frac{1}{100} \right) + \ldots + d_k \left( \frac{1}{10^{k-1}} \right) + \ldots
\]

\[
10m = d_1 n + r_1 \quad \frac{10m}{n} = d_1 + f_1 \quad f_1 = \frac{r_1}{n} = d_2 \left( \frac{1}{10} \right) + \ldots
\]

\[
10r_1 = d_2 n + r_2 \quad \frac{10r_1}{n} = d_2 + f_2 \quad f_2 = \frac{r_2}{n} = d_3 \left( \frac{1}{10} \right) + \ldots
\]

\[
\vdots
\]

\[
10r_{k-1} = d_k n + r_k \quad \frac{10r_{k-1}}{n} = d_k + f_k \quad f_k = \frac{r_k}{n} = d_{k+1} \left( \frac{1}{10} \right) + \ldots
\]

\[
\vdots
\]

Each of \( r_1, r_2, \ldots, r_k, \ldots \) \in \{0, 1, \ldots, n - 1\}
The Algorithm of Successive Multiplications by 10 and Divisions by $n$

- Can this procedure terminate?
The Algorithm of Successive Multiplications by 10 and Divisions by $n$

- Can this procedure terminate?
- yes, when $r_k = 0$.
The Algorithm of Successive Multiplications by 10 and Divisions by $n$

- Can this procedure terminate?
- Yes, when $r_k = 0$.
- If not, $\{d_i, r_i\}$ starts repeating.
Proof of Terminating Sequences using Successive Multiplications and Divisions

\[
\frac{10m}{n} = d_1 + d_2\left(\frac{1}{10}\right) + d_3\left(\frac{1}{100}\right) + \ldots + d_k\left(\frac{1}{10^{k-1}}\right) + \ldots
\]

\[
10m = d_1 n + r_1 \quad \frac{10m}{n} = d_1 + f_1 \quad f_1 = \frac{r_1}{n} = d_1 + d_2\left(\frac{1}{10}\right) + \ldots
\]

\[
10r_1 = d_2 n + r_2 \quad \frac{10r_1}{n} = d_2 + f_2 \quad f_2 = \frac{r_2}{n} = d_2 + d_3\left(\frac{1}{10}\right) + \ldots
\]

\[
10r_{k-1} = d_k n + 0 \quad \frac{10r_{k-1}}{n} = d_k + f_k \quad f_k = 0
\]

Algorithm stops at \( k : r_k = 0 \) implies:

\( r_{k+1} = r_{k+2} = \ldots = 0 \) and \( d_{k+1} = d_{k+2} = \ldots = 0 \).

\[
\frac{m}{n} = 0.d_1d_2\ldots d_k.
\]
Examples of fractions with terminating decimal representation

1. \( \frac{m}{n} = \frac{1}{4}, \ m = 1, \ n = 4 \)

\[
10 \times 1 = 2 \times 4 + 2 \iff \frac{10 \times 1}{4} = 2 + \frac{2}{4}, \ (d_1 = 2, \ r_1 = 2) \\
10 \times 2 = 5 \times 4 + 0 \iff \frac{10 \times 2}{4} = 5 + \frac{0}{4}, \ (d_2 = 5, \ r_2 = 0)
\]

\( r_2 = 0 \) implies \( \frac{1}{4} = 0.d_1d_2 = 0.25 \)
Examples of fractions with terminating decimal representation

2. \( \frac{m}{n} = \frac{5}{8}, m = 5, n = 8 \)
Examples of fractions with terminating decimal representation

2. \( \frac{m}{n} = \frac{5}{8}, m = 5, n = 8 \)

\[
\begin{align*}
10 \times 5 &= 6 \times 8 + 2 \iff \frac{10 \times 5}{8} = 6 + \frac{2}{8}, (d_1 = 6, r_1 = 2) \\
10 \times 2 &= 2 \times 8 + 4 \iff \frac{10 \times 2}{8} = 2 + \frac{4}{8}, (d_2 = 2, r_2 = 4) \\
10 \times 4 &= 5 \times 8 + 0 \iff \frac{10 \times 4}{8} = 5 + \frac{0}{8}, (d_3 = 5, r_3 = 0)
\end{align*}
\]

\( r_3 = 0 \) implies \( \frac{5}{8} = 0.d_1d_2d_3 = 0.625 \)
Successive Multiplications and Divisions: Non Terminating Representations

\[ \frac{10m}{n} = d_1 + d_2 \left( \frac{1}{10} \right) + d_3 \left( \frac{1}{100} \right) + \ldots + d_k \left( \frac{1}{10^{k-1}} \right) + \ldots \]

Each of \( r_1, r_2, \ldots, r_k, \ldots \in \{1, \ldots, n - 1\} \) and \( r_i \neq 0 \) for all \( i \).
Second tool: Use of Pigeon hole Principle in proving that Infinite representations for $\frac{m}{n}$ have repeating patterns

**Statement:**

If you have $n$ pigeons to occupy $n - 1$ holes:

Then **at least 2 pigeons must occupy the same hole.**
Example 10 pigeons and 9 pigeon holes
Example of 3 pigeons and 2 pigeon holes
Solution of example of 3 pigeons and 2 pigeon holes

OR

Rational numbers vs. Irrational numbers
Application of Pigeonhole Principle for non-terminating sequences

\[
10m = d_1 n + r_1 \quad \Leftrightarrow \quad \frac{10m}{n} = d_1 + \frac{r_1}{n}
\]
\[
10r_1 = d_2 n + r_2 \quad \Leftrightarrow \quad \frac{10r_1}{n} = d_2 + \frac{r_2}{n}
\]
\[
\vdots
\]
\[
10r_{k-1} = d_k n + r_k \quad \Leftrightarrow \quad \frac{10r_{k-1}}{n} = d_k + \frac{r_k}{n}
\]
\[
\vdots
\]

\[
r_1 = \quad r_2 = \quad \ldots \ldots \quad r_{n-1} = \quad r_n =
\]

\[
1 \quad 2 \quad 3 \quad n-3 \quad n-2 \quad n-1
\]

By Pigeonhole principle: At least two remainders \( r_j, r_k, \)
\[
1 \leq j < k \leq n: \quad r_j = r_k.
\]
Applying the Pigeon hole Principle to obtain repeating sequences

Let \{j, k\} be the first pair, such that:
1 ≤ j < k ≤ n and \(r_j = r_k\) then:

\[10r_j = d_{j+1}n + r_{j+1} \quad \text{and} \quad 10r_k = d_{k+1}n + r_{k+1}\]

\[\downarrow\]

\[d_{j+1} = d_{k+1} \quad \text{and} \quad r_{j+1} = r_{k+1}…\]

More generally,

\[d_{j+l} = d_{k+l} \quad \text{and} \quad r_{j+l} = r_{k+l}, \quad 1 ≤ l ≤ k - j.\]

and therefore by recurrence:

\[\frac{m}{n} = 0.d_1d_2…d_j\underbrace{d_{j+1}…d_k}_{}\]
Example

\[ f = \frac{m}{n} = \frac{6}{7} \]

<p>| | | |</p>
<table>
<thead>
<tr>
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<tbody>
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<td>10 \times 6 = 8 \times 7 + 4</td>
<td>d_1 = 8</td>
<td>r_1 = 4</td>
</tr>
<tr>
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<td>d_2 = 5</td>
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<tr>
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<td>10 \times 1 = 1 \times 7 + 3</td>
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</tr>
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</tr>
<tr>
<td>10 \times 6 = 8 \times 7 + 4</td>
<td>d_7 = 8</td>
<td>r_7 = 4</td>
</tr>
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</tbody>
</table>

Each of \( r_1, r_2, r_3, r_4, r_5, \ldots \in \{1, 2, 3, 4, 5, 6\} \).
Example \( f = \frac{m}{n} = \frac{6}{7} \)

\( r_1 = 4 \), \( r_2 = 5 \), \( r_3 = 1 \), \( r_4 = 3 \), \( r_5 = 2 \), \( r_6 = 6 \), \( r_7 = 4 \)

\( \{1, 7\} \) is the first pair, such that \( r_1 = r_7 \) then:

\[
\frac{6}{7} = 0.d_1d_2d_3d_4d_5d_6d_7 = 0.8571428
\]

Length of pattern is 6.
Exercise

Find the decimal representation of

\[ f = \frac{m}{n} = \frac{2}{3} \]

using Successive Multiplications and Divisions
Solution of the exercise \( f = \frac{m}{n} = \frac{2}{3} \)

<table>
<thead>
<tr>
<th>10 \times 2 = 6 \times 3 + 2</th>
<th>d_1 = 6 \quad r_1 = 2</th>
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</tr>
<tr>
<td>\vdots \quad \vdots \quad \vdots</td>
<td>\quad \vdots \quad \vdots \quad \vdots</td>
</tr>
</tbody>
</table>

\{1, 2\} is the first pair, such that \( r_1 = r_2 \) and therefore:

\[
\frac{2}{3} = 0.d_1\overline{d_2} = 0.6\overline{6}
\]

Length of pattern is 1

Rational numbers vs. Irrational numbers
Answer to the Main question of Module

\[ \mathcal{R} = \{ \text{Rational Numbers} f, \ 0 < f < 1 \} \]
\[ \mathcal{I} = \{ \text{Irrational Numbers} f, \ 0 < f < 1 \} \]
\[ S = \mathcal{R} \cup \mathcal{I} \text{ with } \mathcal{R} \cap \mathcal{I} = \emptyset \text{ empty set.} \]

**Question:** If we pick at random a number \( f \) between 0 and 1, what is the probability that this number be rational: \( f \in \mathcal{R} \)?
Both $\mathcal{R}$ and $\mathcal{I}$ are **Infinite sets**.
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- $|\mathcal{R}| = \infty_1$ and $|\mathcal{I}| = \infty_2$

Rational numbers vs. Irrational numbers
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- Which one of these two infinities is bigger?

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- $f = 0.d_1d_2..d_k$ or
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Hence, “much more” ways to obtain elements in \( \mathcal{I} \) than in \( \mathcal{R} \).
\( \mathbb{R} \) is “countably infinite”
\( R \) is “countably infinite”

To understand this concept, define for \( n = 1, 2, 3, 4, \ldots \):

\[
R_n = \left\{ \frac{m}{n + 1} \mid m = 1, 2, \ldots, n, \gcd(m, n + 1) = 1 \right\}.
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$$\mathcal{R}_n = \left\{ \frac{m}{n + 1} \middle| m = 1, 2, \ldots, n, \gcd(m, n + 1) = 1 \right\}.$$

Examples of $\mathcal{R}_n$:

$n = 1: \mathcal{R}_1 = \left\{ \frac{1}{2} \right\} = \{r_1\}$

$n = 2: \mathcal{R}_2 = \left\{ \frac{1}{3}, \frac{2}{3} \right\} = \{r_2, r_3\}$

$n = 3: \mathcal{R}_3 = \left\{ \frac{1}{4}, \frac{3}{4} \right\} = \{r_4, r_5\}$

$n = 4: \mathcal{R}_4 = \left\{ \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5} \right\} = \{r_6, r_7, r_8, r_9\}$
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- Examples of \( \mathcal{R}_n \):
  - \( n = 1 \) : \( \mathcal{R}_1 = \left\{ \frac{1}{2} \right\} = \{ r_1 \} \)
  - \( n = 2 \) : \( \mathcal{R}_2 = \left\{ \frac{1}{3}, \frac{2}{3} \right\} = \{ r_2, r_3 \} \)
  - \( n = 3 \) : \( \mathcal{R}_3 = \left\{ \frac{1}{4}, \frac{3}{4} \right\} = \{ r_4, r_5 \} \)
  - \( n = 4 \) : \( \mathcal{R}_4 = \left\{ \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5} \right\} = \{ r_6, r_7, r_8, r_9 \} \)
  - Check \( n = 5 \) : \( \mathcal{R}_5 = \left\{ \frac{1}{6}, ? \right\} \)
\[\mathcal{R}\] is “countably infinite”

- To understand this concept, define for \(n = 1, 2, 3, 4, \ldots:\)

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- Examples of \(\mathcal{R}_n:\)
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- Check \(n = 5:\) \(\mathcal{R}_5 = \{\frac{1}{6}, ?\}\)

- \(\mathcal{R}_5 = \{\frac{1}{6}, \frac{5}{6}\} = \{r_{10}, r_{11}\}\)
As a consequence, we can enumerate the elements of $\mathcal{R}$:

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$$\mathcal{R} = \{r_1, r_2, r_3, r_4, \ldots\}$$

Implying:

Countable infinity of $\mathcal{R}$ $\iff$ a one to one relation between $\mathcal{R}$ and the natural integers: $\mathbb{N} = \{1, 2, 3, 4\ldots\}$
On the other hand, $\mathcal{I}$ is “uncountably” infinite.
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This follows from the fact that $f$ is irrational if and only if its infinite representation $0.d_1d_2...d_k...$ has all its elements belonging randomly to the set $\{0, 1, 2, ...9\}$. 
On the other hand, $\mathcal{I}$ is “uncountably” infinite.

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At that point, the proof of uncountability of $\mathcal{I}$ can be obtained using Cantor’s proof by contradiction.
Let us assume “countability of $\mathcal{I}$”, i.e. its elements can be listed as \( \{i_1, i_2, i_3, \ldots \} \), a set in a one-one relation with the set of natural numbers.
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\[
i_1 = 0.f_{1,1}f_{1,2}\ldots f_{1,k} \ldots
\]
\[
i_2 = 0.f_{2,1}f_{2,2}\ldots f_{2,k} \ldots
\]
\[
\vdots
\]
\[
i_m = 0.f_{m,1}f_{m,2}\ldots f_{m,k} \ldots
\]

Contradiction $i \in \mathcal{I}$ but different from each of the elements in \( \{i_1, i_2, i_3, \ldots \} \).
Let us assume “countability of $\mathcal{I}$”, i.e. its elements can be listed as $\{i_1, i_2, i_3, \ldots\}$, a set in a one-one relation with the set of natural numbers.

\[
i_1 = 0.f_{1,1}f_{1,2} \ldots f_{1,k} \ldots \\
i_2 = 0.f_{2,1}f_{2,2} \ldots f_{2,k} \ldots \\
\vdots \\
i_m = 0.f_{m,1}f_{m,2} \ldots f_{m,k} \ldots \\
\]

Let $\bar{i} = 0.f_{i,1}, f_{i,2}, \ldots, f_{i,k} \ldots$, such that the $\{f_{i,i}\}$’s are randomly chosen with:

\[
\bar{f}_{1,1} \neq f_{1,1}, \bar{f}_{2,2} \neq f_{2,2}, \ldots, \bar{f}_{k,k} \neq f_{k,k}, \ldots
\]
Let us assume “countability of $I$”, i.e. its elements can be listed as $\{i_1, i_2, i_3, \ldots\}$, a set in a one-one relation with the set of natural numbers.

\[
i_1 = 0.f_{1,1}f_{1,2} \ldots f_{1,k} \ldots
\]
\[
i_2 = 0.f_{2,1}f_{2,2} \ldots f_{2,k} \ldots
\]
\[
\vdots
\]
\[
i_m = 0.f_{m,1}f_{m,2} \ldots f_{m,k} \ldots
\]
\[
\vdots
\]

Let $\bar{i} = 0.\bar{f}_{1,1}, \bar{f}_{2,2}, \ldots, \bar{f}_{k,k} \ldots$, such that the $\{\bar{f}_{i,i}\}$’s are randomly chosen with:

$\bar{f}_{1,1} \neq f_{1,1}, \bar{f}_{2,2} \neq f_{2,2}, \ldots, \bar{f}_{k,k} \neq f_{k,k}, \ldots$

**Contradiction:** $\bar{i} \in I$ but $\bar{i}$ different from each of the elements in $\{i_1, i_2, i_3 \ldots\}$. 

Rational numbers vs. Irrational numbers
Answer to Main Question
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• $|\mathcal{R}| = \infty_1 \equiv \aleph_0$. 

Rational numbers vs. Irrational numbers
Answer to Main Question

- $|\mathcal{R}| = \infty_1 \equiv \aleph_0$.
- $|\mathcal{I}| = \infty_2 \equiv \mathcal{C}$.
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- $|\mathcal{R}| = \infty_1 \equiv \aleph_0$.
- $|\mathcal{I}| = \infty_2 \equiv \mathcal{C}$.
- With $\aleph_0 \ll$ (“much less than”) $\mathcal{C}$. 

Rational numbers vs. Irrational numbers
Answer to Main Question

- $|\mathcal{R}| = \infty_1 \equiv \aleph_0$.
- $|\mathcal{I}| = \infty_2 \equiv \mathcal{C}$.
- With $\aleph_0 <<$ ("much less than") $\mathcal{C}$.

\[ \rightarrow \text{Prob}(f \in \mathcal{R}) = \frac{\aleph_0}{\aleph_0 + \mathcal{C}} \approx \frac{\aleph_0}{\mathcal{C}} \approx 0. \]