• Édouard Lucas
• (1842-1891)

http://faculty.evansville.edu/ck6/bstud/lucas.html
Rules for Moving Disks

1) We can only move one disk at a time.
   We can move it from its current tower to either of the other two towers, assuming we obey Rule #2.

2) Any stack of disks must always be in ascending order, with smallest on the top and largest on the bottom.
Making a Towers of Hanoi – at Home
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<table>
<thead>
<tr>
<th>$N$</th>
<th>$2^N$</th>
<th>$2^N - 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>8</td>
<td>7</td>
</tr>
<tr>
<td>4</td>
<td>16</td>
<td>15</td>
</tr>
<tr>
<td>5</td>
<td>32</td>
<td>31</td>
</tr>
<tr>
<td>6</td>
<td>64</td>
<td>63</td>
</tr>
<tr>
<td>7</td>
<td>128</td>
<td>127</td>
</tr>
<tr>
<td>8</td>
<td>256</td>
<td>255</td>
</tr>
</tbody>
</table>
Sequence of Moves for $N = 3$

• 1. Small to Right.
• 2. Medium to Middle.
• 3. Small to Middle.
• 4. Large to Right.
• 5. Small to Left.
• 6. Medium to Right.
• 7. Small to Right
First Note from Dr. 4 to Dr. 3

• “Please do the $N = 3$ problem, placing the top 3 disks currently on Tower #1 onto Tower #2, the Middle one. Then pass everything back to me.”
Second Note from Dr. 4 to Dr. 3

• “Now please do the $N = 3$ problem again, this time moving the 3 disks on Tower 2 (Middle) onto Tower 3 (Right). Then pass back to me.”
Note from Dr. 5 to Dr. 4

• “Please do the $N = 4$ problem, placing the top 4 disks currently on Tower #1 onto Tower 2, the Middle one. Then pass everything back to me.”
Second Note from Dr. 5 to Dr. 4

• “Please again do the $N = 4$ problem, placing the 4 disks currently on Tower #2 onto Tower #3, the Right one. Then pass everything back to me.”
The Good Drs.
The Good Drs.
Our focus:
Dr. 3, Dr. 4, Dr. 5
Recursion is a method of defining \textit{functions} in which the function being defined is applied within its own definition.
Factorial: An Example of Recursion

• By definition,
  \[ N! = N(N-1)(N-2) \ldots (3)(2)(1) \]

• Recursion with Factorial:
  \[ N! = N(N-1)! \]
• \( \text{Sol}(4,1,3) = \text{Sol}(3,1,2) + \text{Sol}(1,1,3) + \text{Sol}(3,2,3) \).
• Or,
• (Best set of moves to move 4 disks from Tower 1 to Tower 3) equals
• (Best set of moves to move 3 disks from Tower 1 to Tower 2) plus
• Movement of largest disk from Tower 1 to tower 3 plus
• (Best set of moves to move 3 disks from Tower 2 to Tower 3)
The 4-Disk Problem

• We have 4 disks:
  – Small
  – Medium
  – Large
  – Super Large

• $\text{Sol}(4,1,3) = \text{Sol}(3,1,2) \text{ PLUS } \text{Sol}(1,1,3) \text{ PLUS } \text{Sol}(3,2,3)$
Sol(3,1,2) = (Set of moves to move 3 disks from Tower 1 to Tower 2)

• 1. Small to Tower 3.
• 2. Medium to Tower 3.
• 3. Small to Tower 3.
• 4. Large to Tower 2.
• 5. Small to Tower 1.
• 6. Medium to Tower 2.
• 7. Small to Tower 2.

PLUS
Sol(1,1,3) = “Movement of ‘Super Large’ from Tower 1 to Tower 3”

PLUS
Sol(3,2,3) = (Set of moves to move 3 disks from Tower 2 to Tower 3)

- 1. Small to Tower 3.
- 2. Medium to Tower 1.
- 3. Small to Tower 1.
- 4. Large to Tower 3.
- 5. Small to Tower 2.
- 6. Medium to Tower 3.
- 7. Small to Tower 3.
The COUNT Operation

- \( \text{COUNT}\{\text{Sol}(3,1,3)\} = \text{Number of moves required to move 3 disks from Tower 1 to Tower 3.} \)
- \( \text{Sol}(3,1,3) = \)
  
  - 1. Small to Right.
  - 2. Medium to Middle.
  - 3. Small to Middle.
  - 4. Large to Right.
  - 5. Small to Left.
  - 6. Medium to Right.
  - 7. Small to Right

\( \text{COUNT}\{\text{Sol}(3,1,3)\} = 7. \)
Counting Moves

\[
\text{COUNT}\{\text{Sol}(4,1,3)\} = \\
\text{COUNT}\{\text{Sol}(3,1,2)\} + \\
\text{COUNT}\{\text{Sol}(1,1,3)\} + \\
\text{COUNT}\{\text{Sol}(3,2,3)\}.
\]
Counts: Using our Hypothesis

• \( \text{COUNT}\{\text{Sol}(N+1,1,3)\} = \text{COUNT}\{\text{Sol}(N,1,2)\} + \text{COUNT}\{\text{Sol}(1,1,3)\} + \text{COUNT}\{\text{Sol}(N,2,3)\}, \) or

• \( \text{COUNT}\{\text{Sol}(N+1,1,3)\} = \]
\[ 2 \cdot 2^N + 1 - 2 = 2^{N+1} - 1. \]
Proof by Induction
Final Challenge Problem

• $N = 64$ disks.
• Monks correctly moving one disk per second, every second.
• How long for them to complete this **Towers of Hanoi** problem?
  • One day, one week, one month, one year, ten years, 100 years, or longer?
Examples of Trail Markers

http://www.dwelement.com/userfiles/image210_lg.jpg
http://blog2.bibleplaces.com/uploaded_images/c9a5b936cd51_9237/IsraelTrailmarkeratMachteshRamontb110702007.jpg