Flaws of Averages

BLOSSOMS Module

Supplement to Flaw of Averages #2:

“The function of the average is not always the same as the average of the function”

Contents:

The Cookie Example

The example in our video segment of the Flaw of Averages that “the function of the average is not always the same as the average of the function” is written out below:

Dan presents Rhonda with the choice between two plates of cookies. Each plate has two carefully shaped cookies so that their shapes are perfect circles. Dan tells Rhonda that

1. Plate A has two cookies that have an average diameter of 7 cm.
2. Plate B has two cookies that have an average diameter of 8 cm.

Dan then asks Rhonda: “Which plate would you like to have if your goal is to have the largest cookie **area**?”

The initial reaction that we expect your students to have is that they will say that ‘Plate B’ should be Rhonda’s choice. What the example on the video illustrates is that ‘Plate B’ is not guaranteed to give you two cookies with the largest **area**.

For the exact numbers:

Plate A has a cookie of diameter 1 cm and a cookie of diameter 13 cm, so the average diameter is 7 cm and the total cookie area is 42.5π cm^2 (which equals 133.5175 cm^2) for an average area per cookie of 21.25π cm^2 (which equals 66.7587 cm^2).

Plate B has two cookies of diameter 8 cm, so the average diameter is 8 cm, but the total cookie area is only 32π cm^2 (which equals 100.53 cm^2) for an average area per cookie of only 16π cm^2 (which equals 50.265 cm^2).

The Math Behind the Cookie Example

If your class is comfortable with the following algebra, one option for you to work on with your students after this segment would be to work out the algebra with them:

Let $f(x) = c*x^2$

Let x_1 and x_2 be two different values of x.

For the average of the function:

$$\frac{f(x_1)+f(x_2)}{2} = \frac{(c*x_1^2)+(c*x_2^2)}{2} = \frac{c}{2}*(x_1^2+x_2^2)$$

For the function of the average:

$$f\left(\frac{(x_1+x_2)}{2}\right) = c*\left(\frac{(x_1+x_2)}{2}\right)^2 = \frac{c}{4}*(x_1^2+2x_1x_2+x_2^2)$$

From here, we can next find when the average of the function equals the function of the average, which it turns out is **only** when x_1 equals x_2:

$$\frac{f(x_1)+f(x_2)}{2} = f\left(\frac{(x_1+x_2)}{2}\right) \text{ when}$$

$$\frac{c}{2}*(x_1^2+x_2^2) = \frac{c}{4}*(x_1^2+2x_1x_2+x_2^2)$$

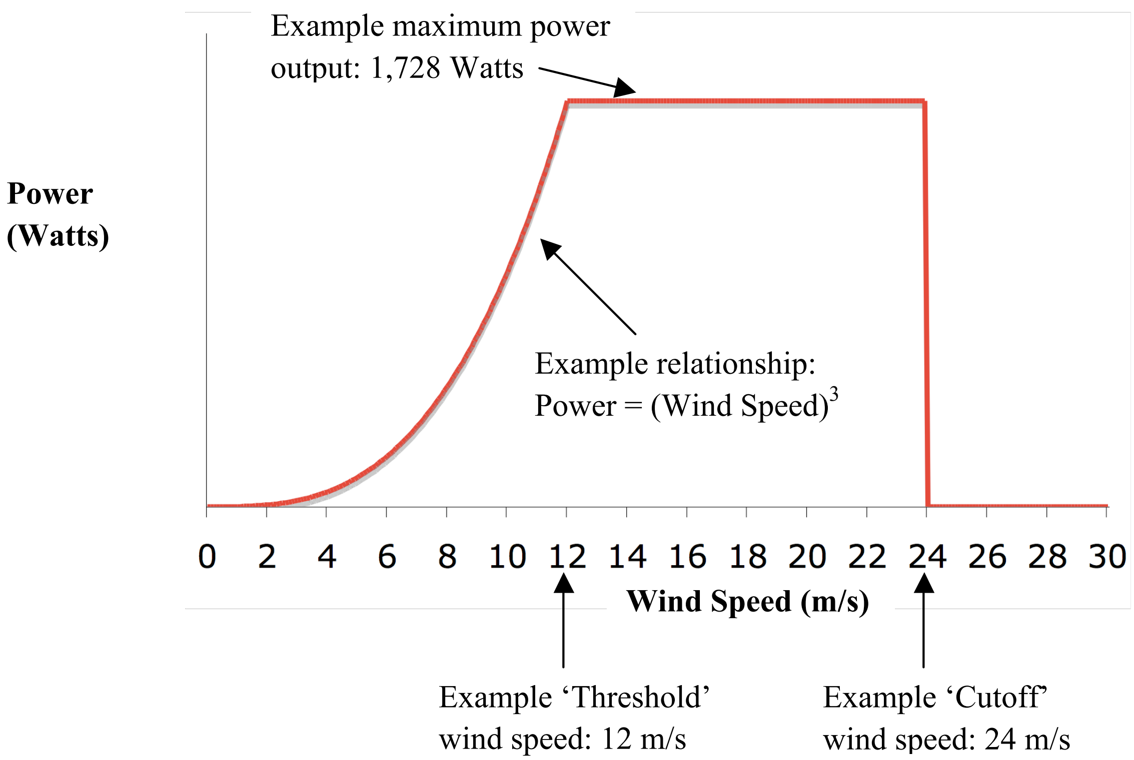
$$\frac{c}{2}*(x_1^2+x_2^2) - \frac{c}{4}*(x_1^2+2x_1x_2+x_2^2) = 0$$

$$\frac{c}{4}*\left(x_1^2 - 2x_1x_2 + x_2^2\right) = 0$$

$$\frac{c}{4}*\left(x_1 - x_2\right)^2 = 0$$

Wind Turbine Example

In many places around the world, wind turbines that generate electricity are popping up on many hillsides and coastlines. The connection between wind turbines and our second Flaw of Averages, "the function of the average is not always the same as the average of the function", is that the power generated by the wind turbine is proportional to the cube of the wind speed when the wind speed is below a certain 'threshold' speed. Beyond this 'threshold' speed, technical limitations in the generator constrain the power output to a fixed level up to a 'cutoff' speed. Beyond the 'cutoff' speed, the wind is blowing too fast for the wind turbine to operate safely, and a combination of braking mechanisms and/or feathering of the turbine blades will be used to prevent the turbine blades from spinning, thus providing no power when the wind is blowing too fast. The figure below illustrates an example relationship between power and wind speed.



We refer you to our online references on the website for more about the mathematical details of the power to wind speed relationship.

Note: The above relationship is a simplified model of the actual physical situation. It is a useful model for quick approximations, though (see the BLOSSOMS module by Stephen Hou on 'The Art of Approximation in Science and Engineering').

With the above example function of

$$\text{Power (in Watts)} = \begin{cases} (\text{Wind Speed})^3 & 0 \ ^{m}\!/_{s} \le \text{Wind Speed} \le 12 \ ^{m}\!/_{s} \\ 1{,}728 & 12 \ ^{m}\!/_{s} < \text{Wind Speed} \le 24 \ ^{m}\!/_{s}, \\ 0 & 24 \ ^{m}\!/_{s} < \text{Wind Speed} \end{cases}$$

we find our Flaw of Averages #2 appearing in a number of scenarios, with two simple examples illustrated here:

<u>Scenario 1: 'Cookie-style' scenario</u>

Site 1A has wind blowing 50% of the time at 0 m/s and 50 % of the time at 10 m/s

Site 1B has wind blowing 100% of the time at 6 m/s

Site 1A average wind speed < Site 1B average wind speed

Site 1A has an average wind speed of 5 m/s

Site 1B has an average wind speed of 6 m/s

Site 1A average power output > Site 1B average power output

Site 1A has an average power output of 500 W ($\frac{1}{2} * 0 + \frac{1}{2} * 10^3 = 500$W)

Site 1B has an average power output of 216 W

Scenario 2: All-or-nothing wind scenario

Site 2A has wind blowing 50% of the time at 0 m/s and 50 % of the time at 26 m/s

Site 2B has wind blowing 100% of the time at 12 m/s

Site 2A average wind speed > Site 2B average wind speed

Site 2A has an average wind speed of 13 m/s

Site 2B has an average wind speed of 12 m/s

Site 2A average power output < Site 2B average power output

Site 2A has an average power output of 0 W

Site 2B has an average power output of 1,728 W