Pythagorean Theorem: Proof and Applications

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Idea

Investigate the history of Pythagoras and the Pythagorean Theorem. Also, have the opportunity to practice applying the Pythagorean Theorem to several problems. Students should analyze information on the Pythagorean Theorem including not only the meaning and application of the theorem, but also the proofs.

1 Motivation

You're locked out of your house and the only open window is on the second floor, 25 feet above the ground. You need to borrow a ladder from one of your neighbors. There's a bush along the edge of the house, so you'll have to place the ladder 10 feet from the house. What length of ladder do you need to reach the window?

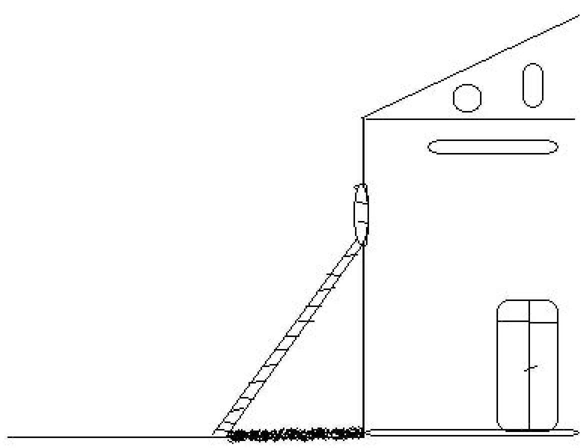

Figure 1: Ladder to reach the window

The Tasks:

1. Find out facts about Pythagoras.
2. Demonstrate a proof of the Pythagorean Theorem
3. Use the Pythagorean Theorem to solve problems
4. Create your own real world problem and challenge the class

2 Presentation:

2.1 General

Brief history: Pythagoras lived in the 500's BC, and was one of the first Greek mathematical thinkers. Pythagoreans were interested in Philosophy, especially in Music and Mathematics?

The statement of the Theorem was discovered on a Babylonian tablet circa $1900 - 1600$ B.C. Professor R. Smullyan in his book 5000 B.C. and Other Philosophical Fantasies tells of an experiment he ran in one of his geometry classes. He drew a right triangle on the board with squares on the hypotenuse and legs and observed the fact the the square on the hypotenuse had a larger area than either of the other two squares. Then he asked, "Suppose these three squares were made of beaten gold, and you were offered either the one large square or the two small squares. Which would you choose?" Interestingly enough, about half the class opted for the one large square and half for the two small squares. Both groups were equally amazed when told that it would make no difference.

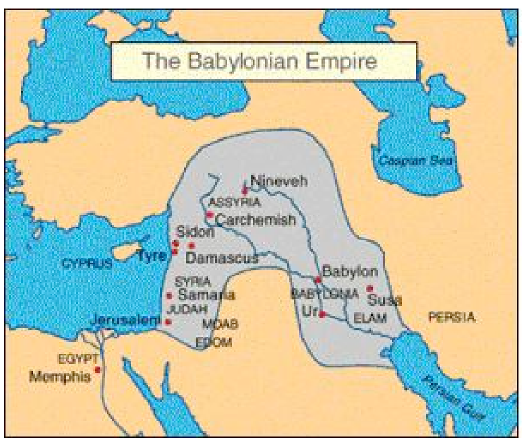

Figure 2: Babylonian Empire

2.2 Statement of Pythagoras Theorem

The famous theorem by Pythagoras defines the relationship between the three sides of a right triangle. Pythagorean Theorem says that in a right triangle, the sum of the squares of the two right-angle sides will always be the same as the square of the hypotenuse (the long side). In symbols: $A^2 + B^2 = C^2$

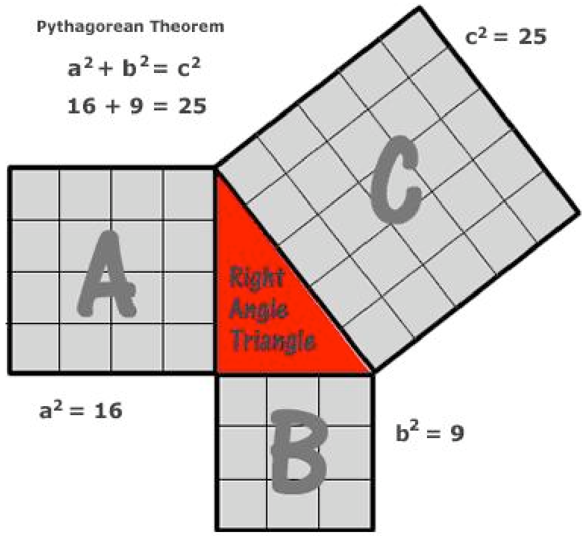

Figure 3: Statement of Pythagoras Theorem in Pictures

2.3 Solving the right triangle

The term "solving the triangle" means that if we start with a right triangle and know any two sides, we can find, or 'solve for', the unknown side. This involves a simple re-arrangement of the Pythagoras Theorem formula to put the unknown on the left side of the equation.

Example 2.1 *Solve for the hypotenuse in Figure* 3.

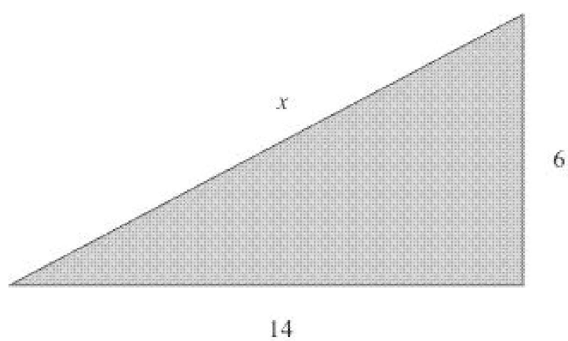

Figure 4: solve for the unknown x

Example 2.2 Applications-An optimization problem *Ahmed needs go to the store from his home. He can either take the sidewalk all the way or cut across the field at the corner. How much shorter is the trip if he cuts across the field?*

2.4 The converse of Pythagorean Theorem

The converse of Pythagorean Theorem is also true. That is, if a triangle satisfies Pythagoras' theorem, then it is a right triangle. Put it another way, only right triangles will satisfy Pythagorean Theorem. Now,

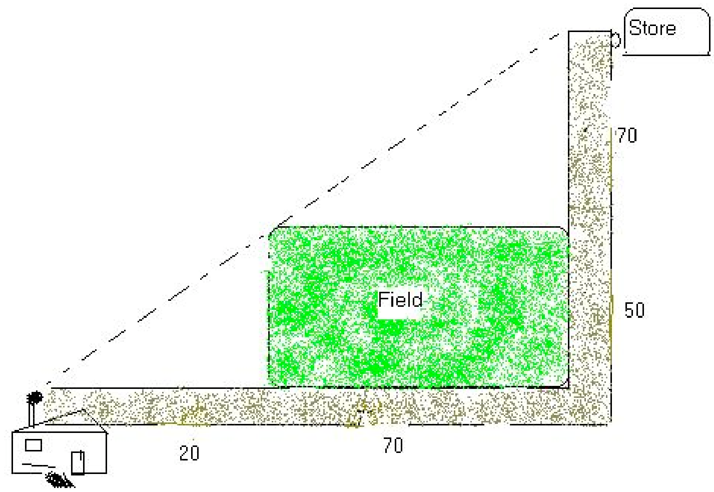

Figure 5: Finding the shortest distance

on a graph paper ask the students to make two lines. The first one being three units in the horizontal direction, and the second being four units in perpendicular (i.e. vertical) direction, with the two lines intersect at the end points of the two lines. The result is right angle. Ask the students to connect the other two ends(open) of the lines to form a right triangle. Measure this distance with a ruler, see Figure 5. Compare with what the Pythagorean Theorem gives.

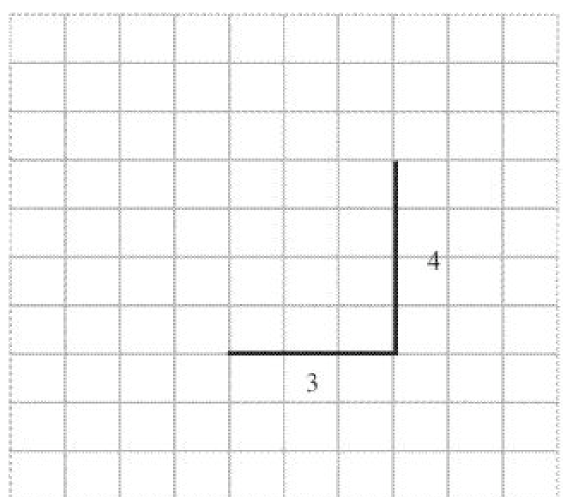

Figure 6: converse of Pythagorean Theorem

2.5 Construction of integer right triangles

It is known that every right triangle of integer sides (without common divisor) can be obtained by choosing two relatively prime positive integers m and n, one odd, one even, and setting $a = 2mn$, $b = m^2 - n^2$ and $c = m^2 + n^2$.

m	n	a	b	c
2	1	4	3	5
3	2	12	5	13
4	1	8	15	17
4	3	24	7	25
5	2	20	21	29
5	4	40	9	41
6	1	12	35	37
7	2	28	45	53
...				...

Table 1: Pythagorean triple

n	$(3n, 4n, 5n)$
2	$(6, 8, 10)$
3	$(9, 12, 15)$
...	...

Table 2: Pythagorean triple

Note that

$$a^2 + b^2 = (2mn)^2 + (m^2 - n^2)^2 = 4m^2n^2 + m^4 - 2m^2n^2 + n^4 = m^4 + 2n^2m^2 + n^4 = (m^2 + n^2)^2 = c^2$$

From Table 1, or from a more extensive table, we may observe

1. In all of the Pythagorean triangles in the table, one side is a multiple of 5.
2. The only fundamental Pythagoreans triangle whose area is twice its perimeter is $(9, 40, 41)$.
3. $(3, 4, 5)$ is the only solution of $x^2 + y^2 = z^2$ in consecutive positive integers.

Also, with the help of the first Pythagorean triple, $(3, 4, 5)$: Let n be any integer greater than 1: $3n$, $4n$ and $5n$ would also be a set of Pythagorean triple. This is true because:

$$(3n)^2 + (4n)^2 = (5n)^2$$

So, we can make infinite triples just using the (3,4,5) triple, see Table 2.

2.6 Proof of Pythagorean Theorem (Indian)

The area of the inner square if Figure 4 is $C \times C$ or C^2,

where the area of the outer square is, $(A + B)^2 = A^2 + B^2 + 2AB$.

On the other hand one may find the area of the outer square as follows:

The area of the outer square = The area of inner square + The sum of the areas of the four **right triangles** around the inner square, therefore

$A^2 + B^2 + 2AB = C^2 + 4\frac{1}{2}AB$, or $A^2 + B^2 = C^2$.

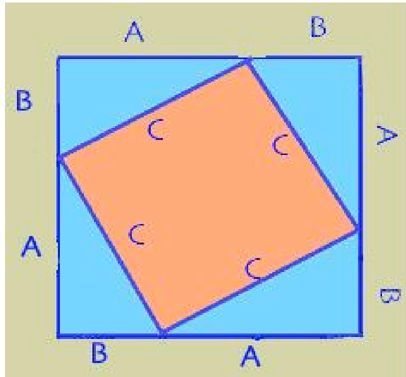

Figure 7: Indian proof of Pythagorean Theorem

2.7 Applications of Pythagorean Theorem

In this segment we will consider some real life applications to Pythagorean Theorem: The Pythagorean Theorem is a starting place for trigonometry, which leads to methods, for example, for calculating length of a lake. Height of a Building, length of a bridge. Here are some examples

Example 2.3 *To find the length of a lake, we pointed two flags at both ends of the lake, say A and B. Then a person walks to another point C such that the angle ABC is* 90. *Then we measure the distance from A to C to be $150m$, and the distance from B to C to be $90m$. Find the length of the lake.*

Example 2.4 *The following idea is taken from [6]. What is the smallest number of matches needed to form simultaneously, on a plane, two different (non-congruent) Pythagorean triangles? The matches represent units of length and must not be broken or split in any way.*

Example 2.5 *A television screen measures approximately* 15 *in. high and* 19 *in. wide. A television is advertised by giving the approximate length of the diagonal of its screen. How should this television be advertised?*

Example 2.6 *In the right figure, $AD = 3, BC = 5$ and $CD = 8$. The angle ADC and BCD are right angle. The point P is on the line CD. Find the minimum value of $AP + BP$.*

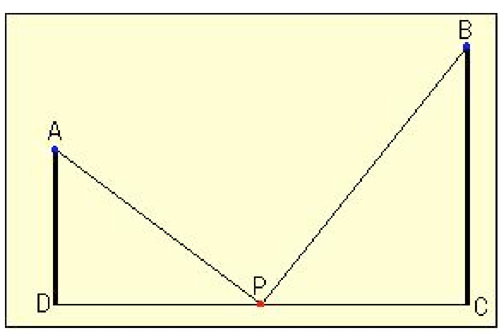

Figure 8: Minimum value of $AP + BP$.

3 Teacher's Guide: Pythagorean Theorem

This module discusses some facts about Pythagorean Theorem. Also, have the opportunity to practice applying the Pythagorean Theorem to several problems. It is suited for students at the 10th grade level. Students should analyze information on the Pythagorean Theorem including not only the meaning and application of the theorem, but also the proofs.

3.1 Teaching Plan

1. Introduction: Introduction establish a common ground between teacher and students, to point out benefits of the use of Pythagorean Theorem in our life, that will lead students to the lesson.

2. Attention: The first step is capturing the student attention either by a puzzle, or a joke (Piece of Gold along each side the triangle).

3. Motivation: Statement given to show why the students need to learn the lesson by showing its importance, a good example is the story of "locked out of your house".

4. Overview: Show the student what to be covered during the class period.

5. Development: Stage of presenting the discussion

   - General and brief history about Pythagorean.
   - Statement of Pythagorean theorem
   - Solving the right triangle
   - Converse of Pythagorean theorem
   - Construction of integer right triangles
   - Proof of Pythagorean theorem
   - Applications of Pythagorean theorem

6. Conclusion: The conclusion should accomplish three things

   - Final summary: Reviews the main points (statement of Pythagorean)
   - Re-motivation : Last chance to let students know why information presented in this lesson are important to the student.
   - Closure: Closure is the signal for lesson end. Like, explain what to do in future, homework exercises. The exercises were written with the assumption that students will use whatever tools (Algebra or Geometry) are available to them.

Finally, it is hoped that this module enables the student to find enjoyment in the study of applications of Pythagorean Theorem in our daily life.

References

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[7] Loomis, Elish Scott, The Pythagorean Proposition: Its Demonstration Analyzed and Classified and Bibliography of Sources for Data of the Four Kinds of 'Proofs', National Council of Teachers of Mathematics,Washington, DC, 1968.