Gilbert Strang's home page is `math.mit.edu/~gs`. The site has links to my video lectures on MIT OpenCourseWare — these are my closest connection to the world of education:

(1) Linear Algebra, (2) Highlights of Calculus, (3) Computational Science

The website also links to my textbooks for these three courses (and to the newest textbook on Differential Equations). The best known book is *Introduction to Linear Algebra*.

My other work as a mathematician is in research and writing and helping the applied mathematics society SIAM. My research combines linear algebra with computational mathematics. My writing includes a number of articles about teaching ideas (these are in the List of Publications and on `web.mit.edu/18.06`). Mathematics is a wonderful life.

Summary and Teacher's Guide

The BLOSSOMS lesson explains how a complex number like $1+i$ is needed and used.

(1) Needed so that $x^2 - 2x + 2 = 0$ has two roots $x = 1 + i$ and $x = 1 - i$

(2) Those numbers are plotted as points in the complex plane

(3) Addition gives $1 + i + 1 - i = 2$, squaring gives $(1 + i)^2 = 1 + 2i + i^2 = 2i$

(4) $1 + i$ is at a 45°angle and $(1 + i)^2 = 2i$ is at a 90°angle with the x-axis.

(5) Angles add when numbers multiply = very important
$(1 + i)^8$ has angle $8 \times 45° = 360°$and $(1 + i)^8 = 16$ (real!)

(6) $x+iy = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta) = re^{i\theta}$ **Euler's great formula**

The lesson can fit in one hour. Two class hours would be best for students to explore the ideas.

I see complex numbers as needed (and perfect) to provide n roots for all polynomials of degree n. Then the many uses of complex numbers take off!

Far beyond polynomials, they are the key to studying all the up-an-down oscillations and all the around-a-circle rotations in nature.

The key to oscillations and rotations is Euler's amazing formula

$$\mathbf{e}^{\mathbf{i}\theta} = \cos\theta + \mathbf{i}\sin\theta.$$

This comes from the fact that angles add when complex numbers are multiplied. On the left side, exponents always add: $(e^2)(e^3) = (e^5)$. It is a natural step to $(e^{2i})(e^{3i}) = (e^{5i})$.

Where are those complex numbers e^{2i}, e^{3i}, e^{5i} in the complex plane??

Those are all on the **unit circle** around zero: radius 1.

Plot them at the angles 2 radians, 3 radians, and 2+3 = 5 radians.

Notice that 5 radians has gone once around the circle and more.

How many radians to go once around or halfway around? 2π and π.

Then Euler gives the fantastic equations $e^{2\pi i} = 1$ and $e^{\pi i} = -1$.

$e^{\pi i} = -1$ contains 6 of the 7 most important symbols in mathematics (#7 is 0).

I don't think this lesson requires special materials. Wikipedia suggests two related ways to explain Euler's great formula $e^{i\theta} = \cos\theta + i\sin\theta$. Compare the infinite series. Solve the differential equation $\frac{dy}{dt} = iy$. The differential equation is what produces the infinite series!

$$y = 1 + it + \frac{1}{2!}(it)^2 + \frac{1}{3!}(it)^3 + \cdots = e^{it}$$

has

$$\frac{dy}{dt} = i + i^2 t + \frac{1}{2!} i^3 t^2 + \cdots = \mathbf{iy} = ie^{it}$$

The one place that outside materials would help is the optional discussion of the Mandelbrot set (the very last topic). Again I start with Wikipedia because it is easily available. The great project would be to write a code that carries out the iteration $z_{n+1} = z_n^2 + z_0$ to see if these numbers approach zero or blow up. This decides whether z_0 is in the Mandelbrot set or not.

A code could explore z_0 near the boundaries of the set, to begin to see the fractal nature and amazing richness of this set.

Let me close by providing a 7-step activity for the breaks. This is for students and teachers together. By including answers, this outline suggests 7 questions. Always good to show complex numbers as points in the complex plane! Plot the points, add and multiply the numbers

1. $x^2 - 4x + 5 = 0$ has roots $x = 2 + i, 2 - i$ and factors $(x - 2 - i)(x - 2 + i)$

2. These roots add to 4 and multiply to 5 (the numbers in $x^2 - 4x + 5$)

3. $2 + i$ is the corner of a right triangle with sides $x = 2, y = 1$, and $r = \sqrt{5}$

4. The angle θ with the x-axis has tangent $= \frac{1}{2}$, cosine $= \frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{5}$, sine $= \frac{\sqrt{5}}{5}$

5. The polar form of $2 + i$ is $r(\cos\theta + i\sin\theta) = \sqrt{5}\left(\frac{2}{\sqrt{5}} + \frac{i}{\sqrt{5}}\right)$

6. For a class that knows addition formulas in trigonometry: **Angles add**

$$(\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi) = \cos(\theta + \phi) + i\sin(\theta + \phi)$$

7. For a class that knows the series for cosine, sine, and e^x :

$$\begin{aligned} \cos x &= 1 - \frac{x^2}{2} + \cdots \text{ and } \sin x = x - \frac{x^3}{6} + \cdots \\ e^{ix} &= 1 + ix + \frac{(ix)^2}{2} + \frac{(ix)^3}{6} + \cdots = \cos x + i\sin x \ \textbf{(Euler)} \end{aligned}$$

I was asked for an assessment tool (not an exam!). I would use parts 1 to 5 with a different equation $x^2 - 6x + 25 = 0$. Then the roots are $3 + 4i$ and $3 - 4i$ and the right triangle has sides 3, 4, 5. To avoid any exam competition, provide the starting point and let students work together. **Enjoy!**