Teacher Guide for the BLOSSOMS Learning Video:

Are Random Triangles Acute or Obtuse?

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This video lesson is a question in "geometrical probability." It has been asked before. A paper by Richard Guy tells something of the history -- which I just learned! (reference to Guy paper below). While mentioning references, I add that a joint paper with Alan Edelman is in preparation and will be available from my website and his. The novel part of that paper will show that the 3:1 obtuse: acute ratio still applies when corners are chosen at random following a normal distribution. (We lose 3:1 if they use a uniform distribution, as in Guy's paper. We recover 3:1 when *angles* are chosen uniformly -- that is the object of this module!)

Here is one more reference, this time to "Bertrand's Paradox," as on Wikipedia. This is about the length of a random chord in a unit circle. The answer again depends on the meaning of random -- three different answers, all completely reasonable.

Teaching from the BLOSSOMS module:

I hope that the triangle drawn on the blackboard, and its physical form as a triangle inside a cube, will become clear as the class thinks about the equation "sum of angles = 180." I chose degrees instead of radians to avoid any unnecessary difficulty (but a remark about this point could be useful). The key idea is that ** a linear equation in 3 dimensions produces a plane**.

Although I have taught linear algebra for (too) many years, I don't have a perfect way to help students see this. Maybe the point is that if two sets of angles add to 180, then their average does too -- geometrically, if two points are on our graph, then so is the point halfway between. This makes the graph flat: a plane.

A valuable discussion to develop with the class: Draw a righttriangle and label edges a,b,c and recall that a^2 + b^2 = c^2. What happens to this equation when the angle grows beyond 90 degrees ?? The class can guess: c^2 is now larger than a^2 + b^2. This is the test for an obtuse triangle. It can be applied in a sampling test of random triangles (after identifying the longest side).

There is an option to continue to the Law of Cosines, which gives the exact difference 2 a b cos(theta) between c^2 and a^2 + b^2. My comments connecting this problem to the more famous Broken Stick problem in Professor Larson's BLOSSOMS module had to be brief -- it is terrific that the two different problems reduce to the same picture: a triangle within a triangle, leading to probabilities 3/4 and 1/4. In both cases we have a linear equation for the sum of the parts -- the pieces of the stick or the angles in my triangle.

In my final segment (not necessarily for the class), I ask about taking three lengths independently -- each can be anywhere from 0 to 1, instead of a known sum for the Broken Stick. Then the question again arises whether the three random lengths can be the sides of a triangle. Those lengths a,b,c, have to satisfy the triangle inequality $a < b + c$ (and two more, $b < a + c$ and $c < a + b$).

The numbers a,b,c give a point in the unit cube. The borderline case $a = b + c$ gives a plane in 3D. That plane cuts off a little pyramid (or tetrahedron, the base is a triangle) from the cube. Its volume is 1/6 (base has area 1/2). Three of those pyramids make volume 1/2, so the chances are 50-50 that a,b,c satisfy the 3 inequalities and make a triangle. Again a nice answer.

References

Lewis Carroll's Obtuse Problem Ruma Falk and Ester Samuel-Cahn - email request to rfalk@cc.huii.ac.il.

Bertrand's Paradox Revisited email request to rclarson@mit.edu.

There Are Three Times as Many Obtuse-Angled Triangles as There Are Acute-Angled Ones, Richard K. Guy, *Mathematics Magazine*, Vol. 66, No. 3 (Jun., 1993), pp. 175-179.