

# Selfish Drivers – Braess’s Paradox and Traffic Planning

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## First Segment:

Dear viewers,

May the peace, mercy and blessing of God Almighty be upon you.

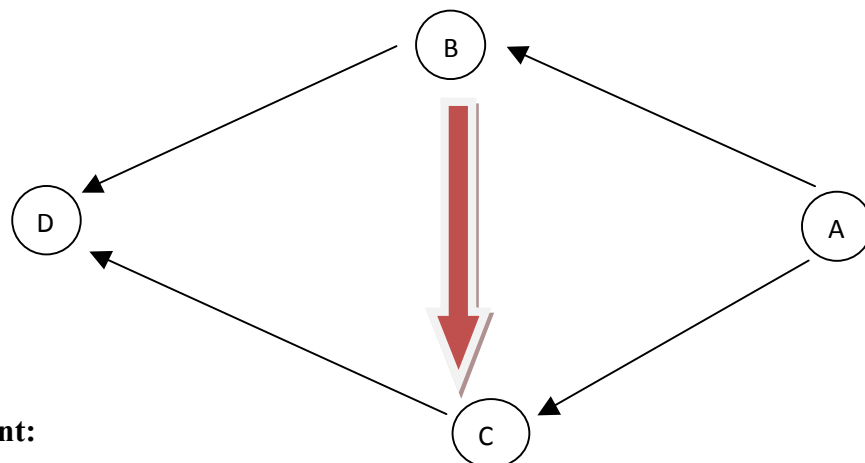
First, I would like to introduce myself. I am Dr. Jawad Younes Abuhlail, from the Department of Mathematics and Statistics, at King Fahd University of Petroleum & Minerals, Dhahran, KSA.

Our topic today is very important. It is primarily related to our personal safety. It is about **selfish drivers**. What is the impact of selfish drivers on traffic congestion, especially inside cities? What is their impact on increasing traffic jams inside cities? What is their impact on increasing car accidents, especially in KSA? This is what we will discuss in this module.

Let’s watch the following scenes. What is the reason for the closure of these roads? Does the closure of these roads impact traffic congestion? Does it increase or decrease traffic jams on the nearby roads? Let’s consider the following scenes. Why do these drivers change their lanes each time? Is there a good reason for each driver who changes his lane? Maybe one of them is forced to turn to a side road and may have an excuse to do so. However, it is noted that many of the drivers, especially the young ones, change their lanes continuously, for example more than once in less than a minute. What is the reason for that? Does this really reduce the time they need to reach their destinations? What is the impact of selfish drivers on traffic? What if all drivers were selfish and not only some of them?

**Activity 1:** Let’s be more specific. We have a number of cars willing to travel from point A to point D and we have the following two paths.

**What is the impact of closing the road BC?**



## Second Segment:

What are the most important factors that determine the behavior of selfish drivers?

It is clear that each selfish driver cares only about his own interest and reducing the time needed to reach his goal regardless the interests of the other drivers.

What do we mean when we say that the roads network is at equilibrium? Equilibrium is reached when each driver realizes that he does not gain any personal benefit from changing his path, and that any change in his path would increase the time he needs to reach his destination. This notion of equilibrium, which is now known as Nash Equilibrium, was introduced by the American scientist John Nash who introduced this notion in his Ph.D. dissertation in 1950 from the last century (and it consisted of 28 pages only).

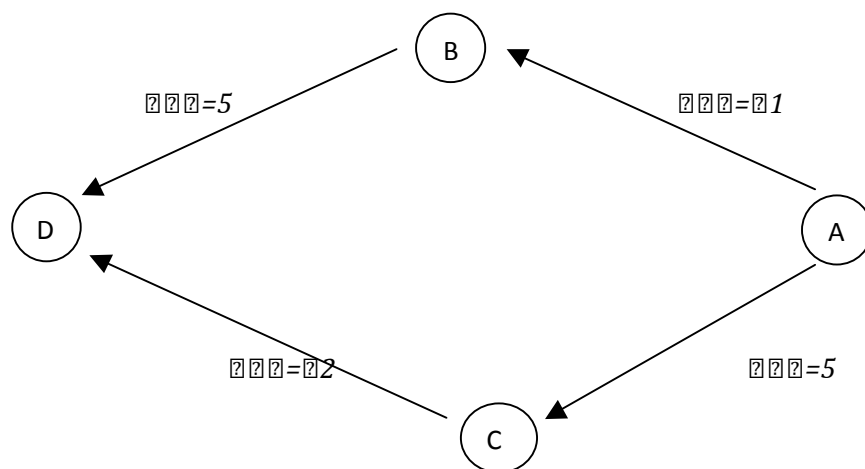
Can this state of equilibrium be always reached? Let us think a little bit. If only some drivers change their paths, we might not reach a state of equilibrium. But if all drivers change their paths each time they think it is appropriate for them, we will reach a state of equilibrium for sure.

What are the factors that affect the time needed to cross part of a road? Condition of the road; the time we cross this road, whether it is in the morning or in the evening.

It is clear that these are important factors, but the most important in determining the time needed to cross a road is the number of cars that cross the road at that time (the same time). This time is given as function  $t = t(x)$  where  $t$  is the time and  $x$  is the number of the cars travelling on the road. All functions that we will be using in the examples that we will discuss later are liner or constant.

**The main question here:** we have a certain number of cars willing to travel from point A to point D. Would closing the road BC decrease or increase the time needed for these cars to travel from A to D?

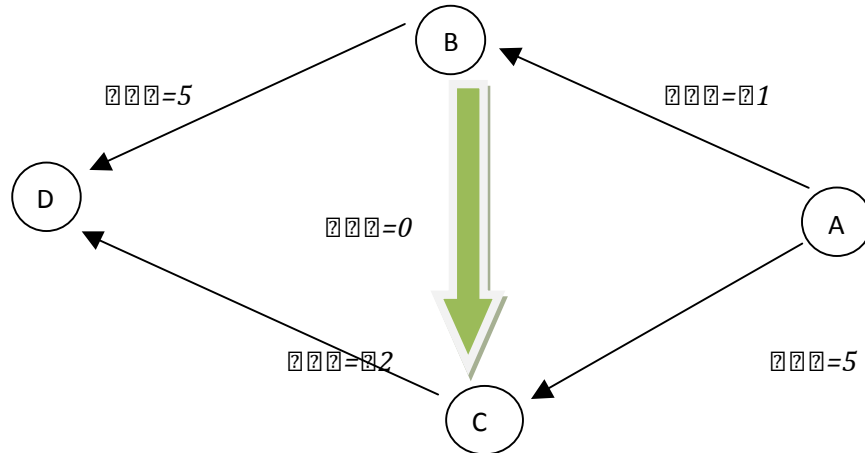
Suppose that we have 4 cars willing to travel from A to D as shown in the figure.



Let the time needed to cross AB be  $t_1$  where  $t_1$  is the number of cars which travels on this road, the time needed to cross CD be  $t_2$  where  $t_2$  is the number of cars which travel on this part of the road,

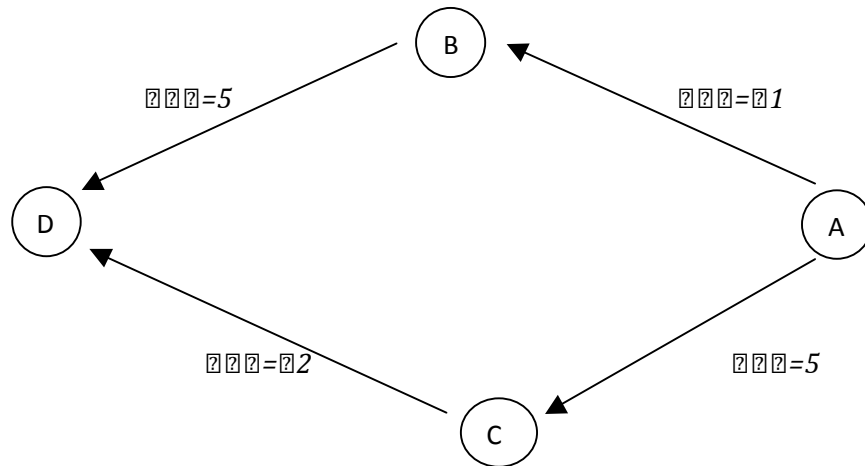
the time needed to cross BD be 5 minutes and the time needed to cross AC be also 5 minutes, by assumption.

**Activity 2:** What if we added a third road from B to C? What is the impact of adding this third road on the time which the four cars need to travel from A to D?



### Third Segment:

We'll try now to answer some of the questions raised about the previous two examples. In the first example and as shown in the diagram



it seems logical to conclude that Nash equilibrium will be reached when two drivers travel through the path ABD while the other two drivers travel through the path ACD. The time that each of the four drivers needs in this case will be 7 minutes as you can double check with your teacher.

**What if we add the road BC?** And let's assume that the time needed to travel from B to C is very short so that it can be neglected.

In this case, there are two points at which the drivers can change their directions: the point A and the point B. Notice that any driver who reaches the point C has no chance to change his path. In this case, we have three paths: ABD, ACD; the third path is ABCD. The time needed to travel on each path is clarified on the screen. Notice that we considered the time needed to travel from B to C to be zero, since it is very short – in seconds – and so can be neglected.

If the first driver decided to go through ACD and then knew that the other three drivers will go through the same path, then he – as well as the others – would need 9 minutes:

$$t_{ACD} = 5 + 2 = 5 + 4 = 9$$

However, if he changed his path to ABD, then he would need 6 minutes:

$$t_{ABD} = 1 + 5 = 1 + 5 = 6$$

But if he gave it a better thought and changed his path to ABCD, then he would need 5 minutes:

$$t_{ABCD} = 1 + 2 = 1 + 4 = 5$$

In this case, and for his own benefit, the first driver will change his path to ABCD.

### Would Nash equilibrium be reached in this case?

Notice that the second driver may change his path. For example, if he went through ABD, then he would need 7 minutes:

$$T_2 = T_1 + 5 = 2 + 5 = 7$$

Notice that  $T_1$  changed in this case from 1 to 2. However, if he changed his path to ABCD, then he would need 6 minutes:

$$T_2 = T_1 + 2 = 2 + 4 = 6$$

and this is the same time which the first driver would also need if the second driver decided to accompany him through ABCD. In this case, and for his personal interest, the second driver would change his path to ABCD.

### Would Nash equilibrium be reached in this case? What if the third and fourth drivers changed their paths too?

If the third driver changed his path to ABD, then he would need 8 minutes:

$$T_3 = T_1 + 5 = 3 + 5 = 8$$

However, if he changed his path to ABCD, then he would need 7 minutes:

$$T_3 = T_1 + 2 = 3 + 4 = 7$$

and this is the same time which the first and second drivers would need if the third driver decided to join them on the path ABCD.

### What about the fourth and last driver? Would he stay on the path ACD?

If he did, then he would need 9 minutes:

$$T_4 = 5 + 2 = 5 + 4 = 9$$

And if he changed his path to ABD, then he would need 9 minutes too:

$$T_4 = T_1 + 5 = 4 + 5 = 9$$

However, if he changed his path to ABCD, then he will need 8 minutes:

$$T_4 = T_1 + 2 = 4 + 4 = 8$$

So, and for his own benefit, and without any consideration of other three drivers' interests, the fourth driver would decide to join them on the path ABCD.

This seems to be illogical, but this is what would happen if all four drivers were selfish.

**Activity 3:** You will be requested to find all possible distributions of the cars on the different paths before and after adding the road BC, and to find the total time needed for the cars to travel from A to D in each case.

**What is the best possible total time?**

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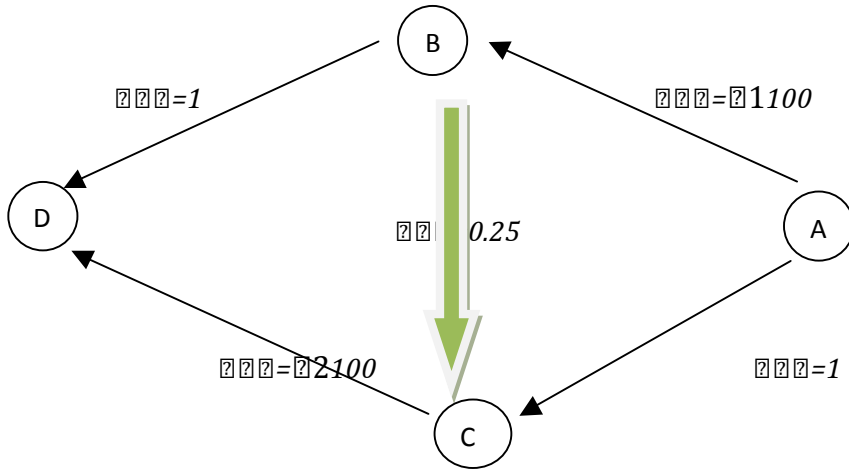
**Activity 4:** Can you find all selfish driving behaviors in the following video segments?

**Fourth Segment:** Samples of selfish drivers' behaviors.

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**Fifth Segment:**

Let us take an additional example. Let's have 100 cars willing to travel from A to D and let us have the roads network shown in the figure



Let  $t_1$  be the number of cars which cross from A to B,  $t_2$  be the number of cars which cross from C to D, where time is measured in hours. Let  $x$  be the number of cars which travel through the path ABD,  $y$  be the number of cars which cross the path ACD and  $z$  be the number of cars which cross the path ABCD.

The time needed to cross ABD is

$$t_{ABD} = t_{AB} + t_{BD} = \frac{1}{100} + 1 = \frac{1}{100} + 1$$

Note that  $t_1 = x + z$ : when we calculate the number of cars which pass through AB, we have to consider the cars which travel through ABCD whose number is  $z$  in addition to the cars whose drivers choose the path ABD and their number is  $x$ .

The time needed to cross ACD is

$$t_{ACD} = t_{AC} + t_{CD} = 1 + \frac{2}{100} = 1 + \frac{2}{100}$$

Note that  $t_2 = z + y$  since the number of cars which cross CD is

The number of the cars through the path ABCD + the number of the cars through the path ACD.

But the time needed to cross the path ABCD is

$$2(1+x) = (x+x+x) + 2x = 1+x+2x100+14$$

**When do we reach the equilibrium?**

We reach the Nash equilibrium when the time needed to cross ABD is equal to the time needed to cross ACD and is equal to the time needed to cross ABCD.

So, we get an equation

$$14+x+x+2x100=1+x+x100=x+x100+1$$

Since in this case, there is no benefit for any driver to change his path as that would increase the number of cars on the new path to which he moved and consequently increase the time he needs to cross the new path.

From that we obtain two equations

$$1+x+x100=x+x100+1 \tag{1}$$

$$14+x+x+2x100=1+x+x100 \tag{2}$$

From the first equation, we have  $x=1$ , and by substituting this in the second equation we get

$$14+2x+2x100=1+x+x100$$

So,  $x+x100=75100$ , i.e.  $x+x=75$ .

Notice that the total number of cars is  $x+(x+x)=100$ ; thus  $x=25$  and this is the same number of cars which cross ACD, i.e.  $y$ . So,  $x=50$  cars.

We conclude that the time needed to cross any of the three paths at equilibrium is one hour and three quarters of an hour.

**Activity 5:** The question which arises now is:

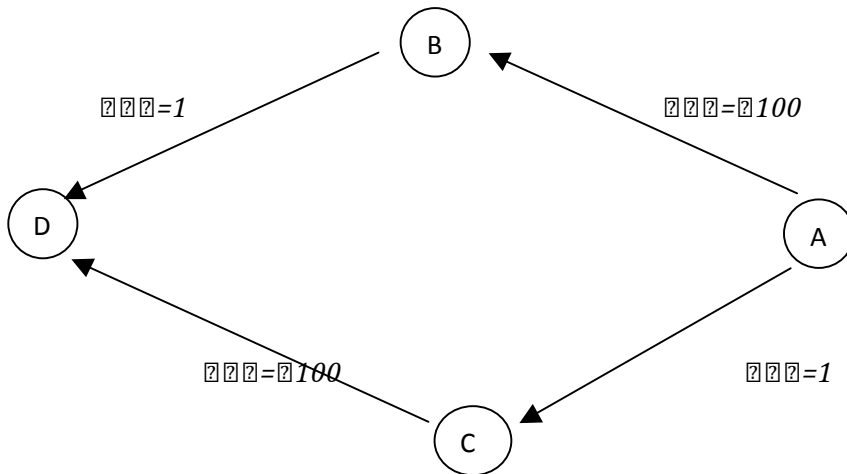
**What is the impact of closing the path BC? When will Nash equilibrium be reached in this case?**

**Sixth Segment:**

**What if we closed the road BC?**



As mentioned in the previous examples, we have only two paths left, namely ABC and the other one is ADC. The shown diagram clarifies these two paths. We also indicated the time needed to cross each part of the roads.



In this case, the time need to cross ABD is

$$2100 + 1 = 2100 + 1$$

while the time needed to cross ACD is

$$1 + 2100 = 1 + 2100$$

It is clear that we reach equilibrium when ... ; very well: when the time needed to cross ABD is equal to the time needed to cross ACD, *i.e.* when

$$2100 + 1 = 1 + 2100$$

whence  $x = 0$  and consequently  $x = 50$  and  $x = 50$  (since the total number of the cars is 100).

Notice that the number of cars which take the path ABCD is zero in this case since there is no road between B and C (we closed this road).

In this case, the time needed to cross from A to D at equilibrium is only one hour and half instead of one hour and three quarter of an hour, which means that closing the road BC saves 15 minutes for each driver at equilibrium.

This is a strange paradox: you close a road and what you get is reducing the time you need to travel from A to D at equilibrium!! This paradox is called Braess's Paradox after the German mathematician Dietrich Braess who introduced this paradox in his Ph.D. dissertation in 1968.

It is a strange paradox as we mentioned earlier. It is a non-trivial famous paradox in Game Theory and simply states:

**Added capacity might degrade performance!!**

**Activity 6:** Let's go back to the previous question.

**Does Braess's Paradox occur in real practice or is it just a matter of theoretical examples which do not have any connection with real practice?**

I will leave you to answer this question with your teachers.

## **Seventh Segment:**

It was noticed that Braess's Paradox applies in several roads networks for example, but not limited to, in Seoul (South Korea), in Stuttgart (Germany) in 1969 and in New York (USA) where closing Road 42 in 1990 resulted in reducing traffic congestion in that area.

The question that arises now is:

**Is it possible to predict that closing specific roads may lead to improvement in traffic flow?**

A study conducted by a group of scientists in 2008 predicted that Braess's paradox might occur in specific paths in Boston, New York and London. They also indicated specific roads, the closure of which might cut down traveling time in these cities.

Another question that arises is:

**Can Braess's paradox be avoided?**

A new study published recently by the American scientist Anna Nagurney in 2010 proved, for the first time, that Braess's Paradox may disappear in some roads networks that are heavily used.

**Does this mean that constructing new roads is always a good thing?**

In the same study, the American scientist clarified that what happens in this case is that drivers do not use the new added roads due to a phenomena which she called **crowd wisdom**. We conclude what the study assures the importance of taking caution while designing the infrastructure for roads since high demand on some roads may lead to completely avoiding using some parts of the network. It is worth mentioning that Braess's Paradox has other applications such as in the internet.

Now we will provide a complete answer to the main question of this module:

**Does adding new roads always result in reducing traffic congestion in road networks?**

The answer can be summarized as follows:

Adding a new road to a road network may sometimes degrade its general performance instead of improving it, and may result, in cases of high demand, complete avoidance of some parts of the network.

This is very strange but it happens.

**Activity 7:** Do you know of any roads in your region the closure of which might improve the traffic flow?

## **Teacher's Guide**

In the name of God the Merciful

Dear teachers,

At the beginning, I hope that you would like this module and find it suitable for your students. I hope also that the suggested class activities are interesting and suitable.

The main objective of this module is raising the awareness about the hazard of selfish drivers and their negative impact on traffic flow especially inside cities, in addition to the possibility of causing traffic accidents.

The main learning objectives of this module can be summarized as follows:

- 1) Providing a real practice from a branch of Mathematics, namely Game Theory
- 2) Encouraging students to conduct scientific research and reminding them about its importance especially in solving problems related to their own countries
- 3) Training students on counting methods and on finding all possible cases in a given example
- 4) Introducing Nash equilibrium in a simple way
- 5) Introducing Braess's Paradox in a simple way

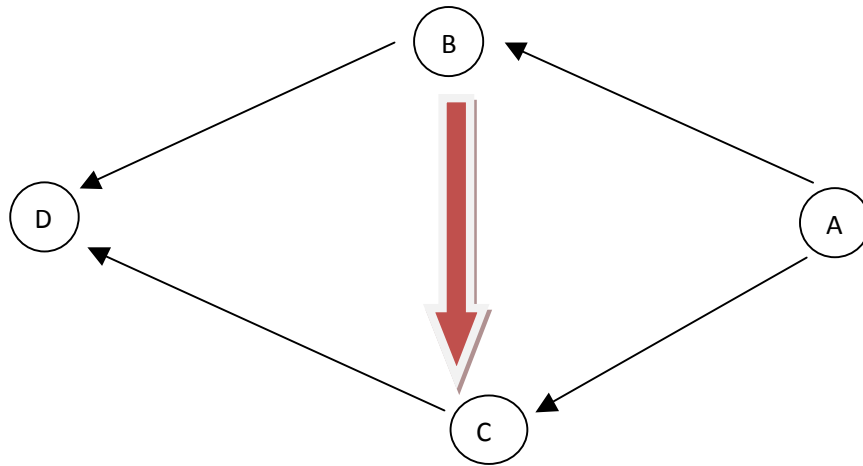
The prerequisites needed to understand this module are summarized as follows: solving two equations in two unknowns by deletion or substitution.

I am pretty sure that any student in the middle school would be able to understand and follow this module.

Seven activities were suggested, one following each of the seven segments in this module. These can be summarized as follows:

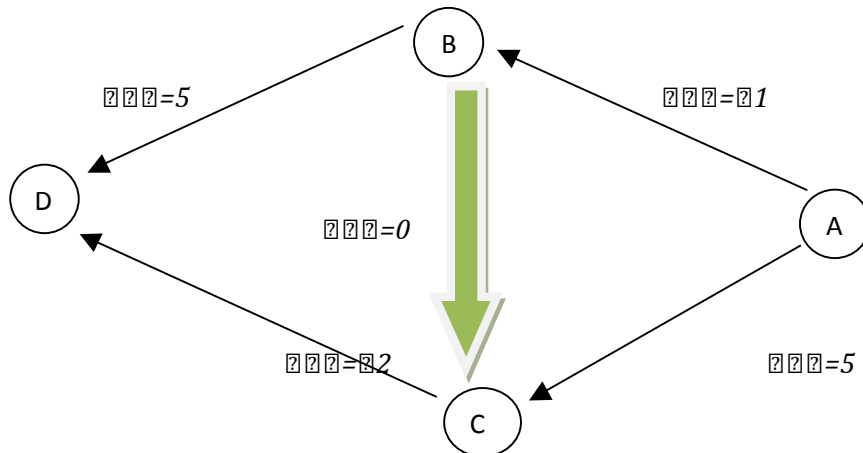
**Activity 1:** The students shall try to answer the main question:

Consider the shown diagram



**Would closing the road BC result in increasing or decreasing the traffic congestion in the nearby roads, i.e. the paths ABD and ACD?**

**Activity 2:** The students shall try to answer the following questions related to the shown diagram



At what points do drivers have the option to change their paths? At what point(s) would the drivers have no chance to change their paths; instead, they are forced to continue through a certain path?

When is Nash equilibrium achieved? What is the time needed for each of the four drivers in the first and the second examples at equilibrium?

**Activity 3:** In the first and the second examples, the students shall try to find all possible distributions of the four cars on the available paths before and after adding the road BC. The students shall also try to find the total time that the four cars need to travel from A to D in each case they find. Moreover, the students shall be requested to find the best possible total time.

**Activity 4:** The students shall discuss the selfish drivers' behaviors in the shown video segments, which were captured in the eastern province of KSA. The students shall discuss the possibility of modifying such selfish behaviors and discuss whether they are logical or illogical, and what the correct behavior in each case is.

**Activity 5:** The students shall discuss the following questions: What is the impact of closing the road BC? When will Nash equilibrium be achieved in this case, *i.e.* after closing the road BC? What is the total time needed to travel from A to D in this case?

**Activity 6:** The students shall discuss the possibility that Braess's paradox occurs in real practice? Would Braess's paradox occur in real practice or is it just based on theoretical examples.

**Activity 7:** The students shall discuss the possibility that some specific roads might exist in their cities, villages or regions, the closure of which might improve the traffic flow.

I would like also to draw your attention dear teachers that you can assign your gifted students or those who are interested in the module's topic some additional activities such as:

- Writing a simple report about Nash equilibrium and its applications in real practice.
- Writing a simple report about Braess's Paradox and its application in real practice.

At the end, I hope you liked this module and found it useful for you and your students.

May the peace, mercy and blessing of God Almighty be upon you.