BLOSSOMS MODULE
ARE RANDOM TRIANGLES ACUTE OR OBTUSE?

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Hi! I'm Gilbert Strang. I teach math at MIT. Mostly what I teach is linear algebra, vectors and matrices. I love that subject! My life is teaching linear algebra to the world. One way to reach a lot of people you might know about is open courseware. MIT made all 2,000 courses freely available on the web at <OCW.mit.edu> for open courseware. And about 12 or 15 of those courses have video lectures of the whole course, and that includes linear algebra. So a lot of people watch them. I hope you'd like them.

My topic today is more geometry than linear algebra. So here's the question that today is aimed to solve: Is a random triangle acute or obtuse?

First, let's remember what those words mean, acute or obtuse. So here's a little picture. There's an acute triangle. All the angles are below 90°, less than 90°. Here's an obtuse triangle where one of the angles is larger than 90°. My question is, which one is more common, which one is more likely? Let me ask you. If you think about a triangle, just a triangle, just visualize one in your mind, it is acute or obtuse? Stick up your hand for acute. Well, I can't see hands but I'm surprised if acute doesn't have the majority. Somebody might think of obtuse, it's certainly possible.

The question is, what fraction of random triangles is acute or obtuse. The tricky word there is random. What's a random triangle? We'll have a few ways to answer that.

I just want to say one thing. What about a right triangle? That's another possibility. Why don't I put that into the mix? I think the odds of a right triangle are zero. I think that the right triangle… maybe you need to think about this. Why do I say that the probability is zero for a right triangle and I'm looking for the probabilities of these two that are positive probabilities. And the question is what are they? Thanks!

Hi again. So I left you with this picture of an acute triangle and an obtuse triangle and I wiped out the right triangle, saying that the probability was zero there. It's like the probability if you take a random decimal, it being exactly the square root of two. I think the right angles are right on 90°, and we don't expect to hit an angle exactly. But what about these two possibilities? So here's our question: acute or obtuse?

I've got at least two ways to go for it. One would be to think of the angles as random. Another way, which could give a different answer, is to think of the corners as chosen at random. Just pick three points, connect them by a triangle and ask, "Is that triangle acute or obtuse?"

So when I started thinking about this, I thought of one approach which now I don't like. But let me say what it was anyway. One way to do it would be start with one edge, we can make it horizontal. And ask the question, "So now I've got two corners, one, two, and where could the third corner be to be acute or obtuse?" If the third corner is like anywhere at random, which areas will give me an acute answer and which will give me obtuse? Well, let me put in a couple of lines here and you can see if the point is up there, that's going to be acute. If I connect that triangle, that's an acute one. So somehow we've got quite a bit of space where that third corner would make an acute triangle. But all over here, if the third corner wasn't there, if instead I put it over there, so that there's the triangle, so this guy was acute, but all over here, everything is obtuse. And over here, this one would come off in this direction. I guess I'm seeing from this picture that obtuse is winning overwhelmingly. I mean there's a lot more space. If the point could be randomly chosen anywhere, the odds are high that it's going to be in these big spaces. And actually if you're really on the ball in geometry you might realize also if the third point is there, then I've got a big angle. So there's a little obtuse region even inside the strip. Maybe if that third point is inside this semi-circle, then that angle will be bigger than 90. So this is an obtuse region of a big acute region, but an even bigger, way, way bigger obtuse region. I don't see how to get a number out of that.

So another way. OK. The question was whether to use random angles and I'm going to start with those. Random angles. Three angles at random. Of course they can't be completely random because they have to add up to… can I give Greek letters to these angles. I don't know why but alpha for that angle, beta for that angle and gamma for that angle. And what do we know? We know that alpha plus beta plus gamma is 180°. Similarly here right? Alpha, beta, gamma, they add up to 180 always. So when I say random angles, I mean random subject to this requirement that they add up to 180.

Now I want to think. How do I get a picture of the possibilities here, the alpha, beta, gamma, that add up to 180? Well, to me that's a linear equation. When I think of a linear equation I think of being able to make a graph, make a picture. OK. So here I've got an alpha direction, how big the alpha angle is. A beta and a gamma. And I want to plot in this 3D space, this sort of angle space, I want to plot all the angles. They have to be greater or maybe equal zero of course and they have to add to 180. One would be 60-60-60 - that would be equilateral. Somehow that would sit here. 60° each way sits out there. But now can you imagine all other possibilities? Well, this is linear algebra. Now remember what you've already seen about linear algebra and then I want to say what this picture looks like and think about it.

What do you remember about linear algebra? Well, you remember equations like $x + y = 4$, that's a linear equation. Now we're only in 2D. Let me draw a little picture. $x + y = 4$ is goes through (4, 0) and (0, 4) - all the points on that line solve that equation. That's what I want you to move up to 3D on. Linear algebra can be ten dimensions, no problem. Linear algebra is great in ten dimensional space. But here is one line and then maybe what you remember, another one like $x – y = 0$ or something. That would be the line $x = y$ going up this way and there would be a meeting point. That's what you remember about linear algebra. Two equations, two unknowns, two lines, one answer. OK.

But now we're up here. What do I get? Final question for this segment, key question for the whole thing: what's the picture of all the points that satisfy alpha plus beta plus gamma equal 180? Let me give you my answer and think about this. This is like some real thinking.

To me that represents a plane, a flat plane in 3D and that plane goes through the point 180 where alpha is 180 and beta and gamma are zero. So sitting right there. It also goes through this point. It also goes through this point. And it's the plane that goes through those three points. Are you visualizing it? I'll have a little demo in the next segment so you can see it. But to me there's a plane there and going through those three points, the origin there is like behind the plane. So here are the edges of the plane and that's a triangle. That's exactly what I'm looking for. Thanks!

Hi! So we're back to this all important triangle. Remember that triangle is filled in. All those points in 3D satisfy this equation. They're all OK triangles, alpha, beta and gamma add to 180. Well they're almost OK. The very edges of the triangle would be like alpha 180, beta and gamma both zero. So you would say that point the triangle is completely squashed. It's like one angle zero, one angle zero, one angle 180. Not much of a triangle. But when I move inside… well, let me stay on the outside but move along a bit. That's a quite important point. That's the point like halfway between here. So I would say that's the point where if gamma is still zero because I'm down on the base, I've got no height here but I'm halfway here, that would be a 90/90/0 triangle. Again, with a zero angle it's a bit squashed. So it's the points inside that I'm really interested in. And to visualize that better, I've got something that's truly three dimensional. Of course the trouble is that the board is only two dimensional.

OK. So this is the alpha, the beta, the gamma direction and now I'm going to put down a plane and I'll put it in just like it's drawn right there. You see it goes through the 180 for gamma, 180 for alpha, 180 for beta, in fact written there is the equation, our all important equation that the sum of the angles is 180. So it's the points in that triangle that I want to pick at random. And when I pick a random point I want to know is it an acute point or is it an obtuse point? What's the triangle that corresponds to a point? Well, if the point is down here somewhere, alpha is big, bigger than 90 certainly in the corner. So there's a region here where all the points correspond to obtuse triangles because they're down here in this corner. Alpha is bigger than 90. But then there's a point where alpha is exactly 90. Those are the right triangles where alpha is exactly 90°. And this would be one. And then if I cut this one in half, where these guys were 90 and 90, I think, I hope you agree with me, that this is the big alpha and that direction alpha is smaller. So big alpha has a certain chance, it's the size of this smaller region compared to the whole one.

But now beta could also be big. So there's also a corner here where the triangle is obtuse because beta is bigger than 90. Well here beta was exactly 90. Somewhere halfway along there beta will be exactly 90. So there will be a line. The wonderful thing about linear algebra is everything is straight. None of that calculus stuff! So there I have an obtuse corner because of beta being big.

And finally I'm going to have the possibility that gamma is big, as big as 180, as big as 150, as big as 120, as big as 90, that would be there. So there is obtuse where gamma is too big. And what's left is the region where they're all below 90 and that's this triangle. That's the inner triangle. So I'm ready to ask you to answer the question. Let me show you that same picture here on our model where you see these pieces. And of course those ones are the ones that I said had zero probability, those are the right triangles where I'm just on the edge between acute and obtuse. Anyway, what's the answer now? What fraction of triangles are acute and what fraction of triangles are obtuse? I'll come back with the answer, but you can see the answer. Thanks!

Hi! We are ready for the answer and maybe you have it. That was the picture of all possibilities and on that picture we drew the possibilities that gave obtuse and the ones where all the angles were below 90 that gave acute. And we showed it in this mockup here. This was our triangle. Let me even lift it out. And there's the inside triangle that's acute. So if all the possibilities have equal chance, what are the odds? Well, these four triangles are all congruent, is that the right word? They're all the same size. And therefore my answer is 3 to 1. Obtuse wins over acute 3 to 1. Three quarters are obtuse, which is different from what we thought of at the very beginning as a typical triangle. Our vision was acute but wrong. It's 3 to 1.

Now when I realized this and then I looked at an earlier module, an earlier Blossoms video, maybe you've seen that one already, it has…I want to just mention some neat thing about math. There's a broken stick module that Professor Larson did beautifully. He actually brought a stick in, a yardstick. He randomly picked a couple of points. There's the random again. And he actually brought in a saw and sawed the stick into three pieces. And his question was not the same as my question. His question was: do those three pieces make any kind of a triangle? The three pieces of that length, that length and that length. Well, to me those three pieces are OK. I think that if I have this one I could bring this one like here, maybe a little more there. I could bring this one down here. I could make a triangle. I could make one because this one was not bigger than the sum of those. But if this random point had been there, no way I can make a triangle out of this side, this side and this side, because those two sides are not going to meet. They can't reach.

So all I wanted to say was, again we have three lengths. Length one, length two, and length three. And what's the equation, what's the condition that they have to satisfy? They're coming out of this stick, so L1 plus L2 plus L3 is 36. And the question was, is one of them bigger than the sum of the two? Is one of them bigger than 18? And do you see that amazingly Professor Larson's question and my question lead to the same picture, the same answer? 3 to 1. Three to one which way? Three to one that you CAN'T form a triangle out of these pieces. This is much more typical, that picture, where one of the L's is like the beta before, is bigger than the sum of the other two, bigger than half of the total. OK. I just couldn't help putting in that comment, that often math ideas, the underlying idea, solves more than one problem.

OK. Now can I take you to a different approach to my problem? Back to acute or obtuse. But instead of taking the three angles at random, when we got a nice 3:1 answer, I'm going to take the corners at random. So a typical corner, so the first x and the first y are some point with coordinates xy right? That's a plane over there and it's the xy plane. And here's a couple of random numbers like 1.2, square root of 5, whatever. In they go. Then another pair of random numbers. And that gives me a side of a triangle. And then a third one.

I don't know if you can tell, I've just invented a totally new math notation because I need a third one and I have a really giant XY. So this is my contribution to the world here, that notation! And we draw the triangle and all six of those numbers are chosen at random. Well again, I have to say what does it mean at random? But when I get them, I make the triangle. I ask acute or obtuse? This one looks barely acute to me. But if xy had been over here a little bit, it would have been obtuse. What are the odds?

Well, that asks a different question by starting with random corners. Let me tell you what happened. I tried these and maybe you can. Use some random number generator to generate six random numbers and then I did it 10,000 times. And I was hoping for 3:1 and I'll tell you, the first 20, 15 came out obtuse. I thought, "Golden!" But in 10,000 it wasn't quite 3:1. Acute was a little more than a quarter, like about 0.26. And obtuse came out about 0.74. And after you do 10,000, then I went to 100,000, you have to accept that's what is happening. Those are not one quarter and three quarters in disguise. It's a different problem. OK. So I don't know if I can leave you to do 10,000 more, but this is a different question. And let me, in a final short comment say something about the answer. Good!

Hi! Just coming back for some final words on the random corners approach. First of all how would I tell acute or obtuse? You remember I picked these points at random using some random command, did it maybe 10,000 times. And I got a triangle and maybe the sides of the triangle are a, b, and c. How do I know, is there a nice way to tell whether it's acute or obtuse? I really don't want to figure out all those angles. I'd rather be able to tell on the basis of the side lengths, because with the side lengths I can find the distance there. So you remember right triangles would have something like $a^2 + b^2 = c^2$. I'm thinking of c as the big side. Let's suppose c is the biggest. Sorry that it's not in my picture. Imagine it is. c is the biggest. Then a right triangle, this Pythagoras, neat fact, really magic fact. Now suppose c is even bigger. Suppose $a^2 + b^2$ is smaller than c^2. What kind of a triangle have we got there? I think that one is obtuse. It's beyond the point. c has got too big, whereas you know what I'm going to say: acute would be when $a^2 + b^2$ is bigger than c^2. So a very acute one would be 60-60-60, a, b and c all equal. That would certainly be acute, 60-60-60. And a^2, b^2, c^2 would all be the same and we'd be in this case. Whereas for a right triangle, where c is there, then I'm in this case. An obtuse c is even bigger. OK. So that's a good test.

Well, I have a final question. What is random mean? Actually that's been the question all along. And the typical command for example in a system like Matlab, rand will produce a random number between zero and one. So X is between zero and one. Y is, all these points are between zero and one. So my triangle is sitting in a square of side one. These numbers came from a case when I had a square of side zero to one in the X's, zero to one in the Y's. All my triangles were in a square. Now that was not necessarily required. That's what the rand command did. But another random picture would be to pick three points randomly inside a circle. Or pick three points randomly inside a hexagon. And all those give different answers in this world of geometrical probability.

OK. And actually I have other questions. Some I know the answer to, some not, but thanks for letting me talk about this one! Thanks!

Hi! Well, that was the module. I just wanted to say hello again, introduce myself again. I'm Gil Strang. I mentioned mIT OpenCourseware because now a million people have looked at linear algebra on open courseware. Linear algebra is 18.06. And a brand new set of lectures is just going up for applied linear algebra, computational linear algebra, computational science. So just to say hello. I feel so fortunate to have a chance to make these videos and even more just to have a life in teaching mathematics. I can't imagine a better job! I hope you enjoy it. You get some students who see the point and get new ideas and I hope this module might suggest a new one.

Just while talking I could mention another problem that they could tackle. This would be random sides. Random sides now. Side 1, side 2, and side 3, are random numbers between 0 and 1. So now we would have a cube, that's meant to be most of a cube. Actually the class could even make a cube. And then points in the cube are now three sides.

The question is: do we have a triangle or not? If side 1 is ¾ and side 2 and side 3 happen to come out ¼, then we can't make a triangle. So again if we're somehow pushed toward the corners of the cube, there will be a cutoff there where points in the corners don't give a real triangle because one side is bigger than the sum of the other two. And there will be points toward the center that do give a real triangle. There's a challenge question! And I think the answer is a half. How is that for a great number? I think half of the points in the cube actually give proper triangles. I'm not looking at acute or obtuse now. Just is it a triangle or is one side bigger than the sum of the other two, so no triangle possible. So anyway, that's a suggested extra problem and extra chance to visualize it by actually creating somehow a cube.

I hope you like this topic in geometrical probability. Linear algebra is a much bigger world. Mathematics is even slightly bigger than linear algebra, and we allow calculus every once in a while. And just enjoy! Thanks!

END OF TRANSCRIPT